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The Nuts and Bolts of Proofs

Third Edition

Antonella Cupillari

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Third Edition

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List of Symbols

Natural or counting numbers: $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

Prime numbers $= \{2, 3, 5, 7, 11, 13, \ldots\}$

Whole numbers $= \mathbf{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$

Integer numbers $= \mathbb{Z} = \{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers $= \mathbb{Q} =$ {numbers of the form a/b with a and b integers and $b \neq 0$}

Irrational numbers = {numbers that cannot be represented as the quotient of two integers}

Real numbers $= \mathbb{R} =$ {all rational and irrational numbers}

Complex numbers $= \mathbb{C} =$ {numbers of the form $a + ib$ with a and b real numbers and i such that $i^2 = -1$}

$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$
$n!$ (read "n factorial") is defined for all $n \geq 0$. By definition $0! = 1$.

The expression $\{x|x$ has a certain property$\}$ gives the description of a set. In this context the symbol "|" is read "such that." All objects that have the required property are called "elements" of the set.

$a \in \mathrm{A}$ a is an element of the set A (See section about sets)

$a \notin A$ a is not an element of the set A (See section about sets)

$A \subseteq B$ The set A is contained (or equal to) in the set B (See section about sets)

$A \cup B$ read "A union B" (See section about sets)

$A \cap B$ read "A intersection B" (See section about sets)

$A' = C(A)$ read "complement of A" (See section about sets)

$|x|$ = absolute value of x = distance from 0 to $x = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$

$lcm(a,b)$ = least common multiple of a and b

$GCD(a,b)$ = greatest common divisor of a and b

Some facts and properties of numbers

Trichotomy Property of Real Numbers. Given two real numbers a and b, exactly one of the following three relations holds true:

1. $a < b$;
2. $a = b$;
3. $a > b$.

An integer number a is **divisible** by a nonzero integer number b if there exists an integer number n such that $a = bn$. The number a is said to be a **multiple** of b, and b is said to be a **divisor** of a.

Numbers that are multiples of 2 are called **even**. Therefore, for any even number a there exists an integer number k such that $a = 2k$. Numbers that are not divisible by 2 are said to be **odd**; thus any odd number t can be written as $t = 2s + 1$ for some integer number s.

A counting number larger than 1 is called **prime** if it is divisible only by two distinct counting numbers, itself and 1. Therefore 1 is not a prime number.

There are two equivalent definitions that are usually employed when dealing with **rational numbers**. The first is the one given above, with rational numbers considered to be the ratio (quotient) of two integers, where the divisor is not equal to zero. When using this definition, it might be useful to remember that it is always possible to represent a rational number as a fraction whose numerator and denominator have no common factors

(relatively prime) (e.g., use 1/3 instead of (−6)/(−18) or 3/9). This kind of fraction is said to be in **reduced form.**

The second definition states that a number is rational if it has EITHER a finite decimal part OR an infinite decimal part that exhibits a repeating pattern. The repeating set of digits is called the **period** of the number. It can be proved that these two definitions are equivalent.

The two definitions used for rational numbers generate two definitions for **irrational numbers.** The first one is the one given above. The second states that a number is irrational if its decimal part is infinite AND does not exhibit a repeating pattern.

The following relations, definitions, and properties are given only for **positive integer numbers.**

1. The $lcm(a,b) =$ **least common multiple** of a and b, call it L, is the smallest multiple that the positive integers a and b have in common. Therefore:
   i. there exist two positive integers n and m such that $L = an$ and $L = bm$;
   ii. if M is another common multiple of a and b, then M is a multiple of L
   iii. $L \geq a$ and $L \geq b$.
2. The $GCD(a,b) =$ **greatest common divisor** of a and b, call it D, is the largest divisor that the positive integers a and b have in common. Therefore
   i. there exist two positive integers s and t such that $a = Ds$ and $b = Dt$; with s and t relatively prime (i.e., having no common factors),
   ii. if T is another common divisor of a and b, then T is a divisor of D.
   iii. $D \leq a$ and $D \leq b$.

Well-Ordering Principle. Every nonempty set of nonnegative integers contains a smallest element.

Some facts and properties of function

Let f and g be two real valued functions. Then it is possible to construct the following functions:

1. $f + g$ defined as $(f + g)(x) = f(x) + g(x)$
2. $f - g$ defined as $(f - g)(x) = f(x) - g(x)$
3. fg defined as $(fg)(x) = f(x)g(x)$
4. f/g defined as $(f/g)(x) = f(x)/g(x)$ when $g(x) \neq 0$
5. $f \circ g$ defined as $f \circ g(x) = f(g(x))$

The domains of these functions will be determined by the domains and properties of f and g.

A function f is said to be:

1. **increasing** if for every two real numbers x_1 and x_2 such that $x_1 < x_2$, it follows that
$$f(x_1) < f(x_2).$$
2. **decreasing** if for every two real numbers x_1 and x_2 such that $x_1 < x_2$, it follows that
$$f(x_1) > f(x_2).$$
3. **nondecreasing** if for every two real numbers x_1 and x_2 such that $x_1 \leq x_2$, it follows that
$$f(x_1) \leq f(x_2).$$
4. **nonincreasing** if for every two real numbers x_1 and x_2 such that $x_1 \leq x_2$, it follows that
$$f(x_1) \geq f(x_2).$$
5. **odd** if $f(-x) = -f(x)$ for all x.
6. **even** if $f(-x) = f(x)$ for all x.
7. **one-to-one** if for every two real numbers x_1 and x_2 such that $x_1 \neq x_2$, it follows that $f(x_1) \neq f(x_2)$.
8. **onto** if for every value y there is at least one value x such that $f(x) = y$.

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Preface

This book is addressed to those interested in learning more about how and why proofs of mathematical statements work, and it has been written keeping in mind the following remark by George Polya. "A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but... if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery."

Since the only background required is the material covered in a first semester Calculus class, most of the statements considered will deal with basic properties of numbers and functions. The fact that a statement might seem "easy" to understand does not imply that proving it will be an effortless task, as Fermat's Last Theorem[1] has so clearly shown!

[1]Fermat's Last Theorem states that it is impossible to find three (nonzero) integer numbers x, y, and z, such that

$$x^n + y^n = z^n$$

when $n \geq 3$. Pierre de Fermat (1601–1665) claimed he had a proof of this statement, but he never published it and it has not been found. Through the years several mathematicians were able to prove the truth of the statement only for some values of the exponent n. Finally Andrew Wiles, with some help from his colleagues and using several results developed since Fermat's time, offered a complete and lengthy proof of the theorem in 1993. Quite a few years and some very advanced mathematical ideas were needed to prove a theorem that can be stated in one sentence!

The purpose of this book is to help the reader to gain a better understanding of the basic logic of mathematical proofs and to become familiar with some of the basic steps needed to construct proofs. Thus, the mathematical statements to be proved have been kept simple with these goals in mind. It is just like learning where the chords are, before being able to play a nice piece of music!

I would like to thank all my students who keep teaching me that there is always one more way to look at things and one more way to explain something.

I would like to thank the following reviewers for their insight and suggestions: Rob Beezer, University of Puget Sound; Andy Miller, University of Oklahoma; David Vella, Skidmore College; and Maria Girardi, University of South Carolina.

To the Reader

The solutions for all the exercises in this book (except for those in the section "Exercises without Solutions") can be found in the back of the book. These solutions should only be used as a guide. Indeed learning to construct proofs is like learning to play tennis. It is useful to have someone teaching us the basics, and it is useful to look at someone playing, but we need to get into the court and play, if we really want to learn.

Therefore we suggest that you, the reader, set aside a minimum time limit for yourself to construct a proof without looking at the solution (as a starting point, you could give yourself one hour, and then adjust this limit to fit your ability). If you do not succeed, read only the first few lines of the proof presented here, and then try again to complete the proof on your own. If you are not able to do so, read a few more lines and try once more. If you need to read the whole proof, make sure that you understand it, and after a few days, try the exercises again on your own.

Be aware of the fact that in several cases it is possible to construct proofs different from the ones presented in the solutions.

Exercises with the symbol (*) require knowledge of calculus and/or linear algebra.

Introduction and Basic Terminology

Have you ever felt that the words *mathematics* and *frustration* have a lot in common? There are many people who do, including, at times, some very good mathematicians. At the beginner's level, the level for readers of this book, this feeling is often the result of the use of an unproductive and often unsystematic (and panicky) approach that leads to hours of unfruitful work. When anxiety sets in, memorization may look like the way to "survival," but memorization without a thorough understanding is usually a poor and risky approach, both in the short and in the long run. It is difficult to recall successfully a large amount of memorized material under the pressure of an exam or a deadline. It is very easy for most of this material to quickly fall into oblivion. The combination of these two aspects will render most of the work done completely useless, and it will make future use of the material very difficult. Moreover, no ownership of the subject is gained.

The construction of airtight logical constructions ("proofs") represents one of the major obstacles that mathematical neophytes face when making the transition to more advanced and abstract material. It might be easy to believe that all results already proved are true and that there is no need to check them or understand why they are true, but there is much to be learned from understanding the proofs behind the results. Such an understanding gives us new techniques that we can use to gain an insider's view of the subject, obtain other results, remember the results more easily, and be able to derive them again if we want to.

To learn how to read and understand proofs (this term will be defined more precisely in the next few paragraphs) already written in a textbook and to learn how to construct proofs on our own, we will proceed by breaking them down into a series of simple steps and looking at the clues that lead from one step to the next. "Logic" is the key that will help us in this process. We will use the words "logic" and "logical" according to the definition suggested by Irving Copi: "Logic is the study of methods and principles used to distinguish good (correct) from bad (incorrect) reasoning."

Before we start, though, we need to know the precise meaning of some of the most common words that appear in mathematics and logic books.

Statement: A *statement* is a sentence expressed in words (or mathematical symbols) that is either true or false. Statements do not include exclamations, questions, or orders. A statement cannot be true and false at the same time, although it can be true or false when considered in different contexts. For example, the statement "No man has ever been on the Moon" was true in 1950, but it is now false. A statement is *simple* when it cannot be broken down into other statements (*e.g.*, "It will rain." "Two plus two equals four." "I like that book."). A statement is *composite* when it contains several simple statements connected by punctuation and/or words such as *and*, *although*, *or*, *thus*, *then*, *therefore*, *because*, *for*, *moreover*, *however*, and so on (*e.g.*, "It will rain, although now it is only windy." "I like that book, but the other one is more interesting." "If we work on this problem, we will understand it better.").

Hypothesis: A *hypothesis* is a statement that it is assumed to be true, and from which some consequence follows. (For example, in the sentence "If we work on this problem, we will understand it better" the statement "we work on this problem" is the hypothesis.) There are other common uses of the word hypothesis in other scientific fields that are considerably different from the one listed here. For example, in mathematics, hypotheses are never tested. In other fields (*e.g.*, statistics, biology, psychology), scientists discuss the need "to test the hypothesis."

Conclusion: A *conclusion* is a statement that follows as a consequence from previously assumed conditions (hypotheses). (For example, in the sentence "If we work on this problem, we will understand it better" the statement "we will understand it better" is the conclusion.) In *The Words of Mathematics*, Steven Schwartzman writes, "In mathematics, the conclusion is the 'closing' of a logical argument, the point at which all the evidence is brought together and a final result obtained."

Definition: A *definition* is an unequivocal statement of the precise meaning of a word or phrase, a mathematical symbol or concept, ending all possible confusion. Definitions are like the soil in which a theory grows, and it is

important to be aware of the fact that mathematicians do not coin new definitions without giving the process a lot of thought. Usually a definition arises in a theory to capture the properties of some concept that will be crucial in the development and understanding of that theory. Therefore, it is difficult to understand and work with results that use technical terms when the definitions of these terms are not clear. This is similar to working with tools we are not sure how to use; to speaking a language using words for which the meaning is not clear. Knowing and understanding definitions will save a lot of time and frustration.

This is not to suggest that definitions should be memorized by rote, without understanding them. It is a good idea to work with new definitions to be sure that their meanings and immediate consequences are clear, so it will be possible to recall them quickly and appropriately. It is easy to fall behind during a lecture when the speaker uses unfamiliar words, and it is easy to miss much of the speaker's argument while either trying to remember the meaning of the technical terms used or losing interest altogether, thus not understanding what is being said. In this situation, conscious or unconscious doubts about one's technical or mathematical abilities creep in, making successful and efficient learning more difficult. Therefore, we should make sure to have a good starting point by having a clear and thorough understanding of all necessary definitions. It is usually helpful to pin down a definition by finding some examples of objects that satisfy it and some examples of objects that do not satisfy it. Do not confuse the two concepts, though; examples are not definitions and cannot replace them.

Proof: A *proof* is a logical argument that establishes the truth of a statement beyond any doubt. A proof consists of a finite chain of steps, each one of them a logical consequence of the previous one. Schwartzman explains that "the Latin adjective *probus* means 'upright, honest,'... The derived verb *probare* meant 'to try, to test, to judge.' One meaning of the verb then came to include the successful result of testing something, so *to prove* meant 'to test and find valid.'... In a deductive system like mathematics, a proof tests a hypothesis only in the sense of validating it once and for all."

Theorem: A *theorem* is a mathematical statement for which the truth can be established using logical reasoning on the basis of certain assumptions that are explicitly given or implied in the statement (*i.e.*, by constructing a proof). The word *theorem* shares its Greek root with the word *theater*. Both words are derived from the root *thea*, which means "the act of seeing." Indeed, the proof of a theorem usually allows us to see further into the subject we are studying.

Lemma: A *lemma* is an auxiliary theorem proved beforehand so it can be used in the proof of another theorem. This word comes from the Greek word

that means "to grasp." Indeed, in a lemma one "grasps" some truth to be used in the proof of a larger result. The proofs of some theorems are long and difficult to follow. In these cases, it is common for one or more of the intermediate steps to be isolated as lemmas and to be proved ahead. Then, in the proof of the theorem we can refer to the lemmas already established and use them to move to the next step. Often the results stated in lemmas are not very interesting by themselves, but they play key roles in the proof of more important results. On the other hand, some lemmas are used in so many different cases and are so important that they are named after famous mathematicians.

Corollary: A *corollary* is a theorem that follows logically and easily from a theorem already proved. Corollaries can be important theorems. The name, which derives from the Latin word for "little garland," underlines the fact that the result stated in a corollary follows naturally from another theorem. The James & James *Mathematics Dictionary* defines a corollary as a "by-product of another theorem."

General Suggestions

The first step, whether we are trying to prove a result or we are trying to understand someone else's proof, consists of clearly understanding what are the assumptions (hypotheses) made in the statement of the theorem and what is the conclusion to be reached. In this way, we are establishing the starting and ending points of the logical process that will take us from the hypotheses to the conclusion. We must understand the meaning of the hypotheses so we can use the full strength of the information we are given, either implicitly or explicitly, to achieve the desired result. It is essential to check all technical words appearing in the statement and to review the definitions of the ones for which the meanings are not clear *and* familiar.

Examples

1. Suppose we are going to prove the following statement:

   If a triangle is equilateral, then its internal angles are equal.

   We start with the following information:

   i. the object is a triangle (explicit information); and
   ii. the three sides have the same length (explicit information from the word *equilateral*).

But what else do we know about triangles; that is, what implicit information do we have? We can use any previously proven result, not only about triangles but also, for example, about geometric properties of lines and angles in general (implicit information). The conclusion we want to reach is that "the internal angles of the triangle are equal." Therefore, it will be extremely important to know the definition of "internal angles of a triangle" as well.

2. Consider the following statement:

The number a is a nonzero real number.

The statement gives the following information:

i. the number a is different from zero (explicit information); and
ii. the number a is a real number (explicit information).

As mentioned in the preceding, the second fact implicitly states that we can use all properties of real numbers and their operation that the book has already mentioned or requires readers to know (implicit information). Sometimes the hypotheses, as stated, might contain nonessential details, which are given for the sake of clarity.

Examples

1. Consider the triangle ABC.
2. Let A be the collection of all even numbers.
3. Let a be a nonzero real number.

The fact that the triangle is denoted as ABC is not significant. We can use any three letters (or other symbols) to name the three vertices of the triangle. In the same way, we can use any letter to denote the collection of all even numbers and a nonzero real number. The most important thing is consistency. If we used the letters A, B, and C to denote the vertices of a triangle, then these letters will refer to the vertices any time they are mentioned in that context, and they cannot be used to denote another object.

Only after we are sure that we can identify the hypothesis and the conclusion and that we understand the meaning of a theorem to be proved can we go on to read, understand, or construct its proof (that is, a logical argument that will establish how and why the theorem we are considering is true). It is important to observe that a mathematical statement to be proved does not exist in a vacuum, but it is part of a larger context; therefore, its proof might change significantly, depending on the material previously

introduced. Indeed, all the results already established and all the definitions already stated as parts of a context can be used in the construction of the proofs of other results in that same context. As this book focuses more on the "nuts and bolts" of proof design than on the development of a mathematical theory, it does not include the construction of a mathematical setting for the material presented. This approach is supposed to provide the reader with the basic tools to use for the construction of proofs in a variety of mathematical settings.

At this point we want to emphasize the difference between the validity of an argument and the truth or falsity of the results of an argument. An argument is *valid* if its hypothesis supplies sufficient and certain basis for the conclusion to be reached. An argument can be valid and reach a false conclusion, as in the following example, in which one of the hypotheses is false.

All birds are able to fly.
Penguins are birds.
Therefore, penguins are able to fly.

An argument can be invalid and reach a true conclusion. Consider the following argument:

Cows have four legs.
Giraffes have four legs.
Therefore, giraffes are taller than cows.

In the example just given, it is clear that the information we have (cows have four legs; giraffes have four legs) does not imply that "giraffes are taller that cows," which is nonetheless a true fact. The only conclusion we could legitimately reach is that giraffes and cows have the same number of legs.

In other cases the possible flaws in the reasoning process are more subtle. Consider the following argument.

If Joe wins the state lottery, he can afford a new car.
Joe did not win the state lottery.
Therefore, Joe cannot afford a new car.

The hypotheses for this argument are: If Joe wins the state lottery, he can afford a new car. Joe did not win the state lottery. The conclusion reached is: Joe cannot afford a new car.

This is an example of incorrect (nonvalid) reasoning. Indeed, Joe did not win the state lottery, so he might not be able to afford a new car (the

conclusion is true). But, on the other hand, Joe might inherit some money (or he might be already wealthy) and he will be able to afford a new car (the conclusion is false, whereas the hypotheses are still true). Thus, the conclusion does not follow logically from the hypotheses, because the hypotheses *do not* state what Joe will do if he does not win the lottery. So, any conclusion we reach in this case is just speculation as it is not the only possible logical conclusion. This is why the reasoning process is not valid.

When we are working on the proof of a statement, we strive for a *sound* proof; a proof that uses valid arguments, under true hypotheses. It is not unusual to be able to construct more than one sound proof for a true statement, especially when the chain of steps required is rather lengthy.

Very often the construction of a sound proof takes considerable time and effort, and usually the first attempts produce little more than "scratch work." Thus, we must be ready to work on several drafts. While doing this, the elegance of the construction is not the most important issue. After the soundness of a proof is established, it is easier to keep working on it to make it flow well and to remove useless details.

Basic Techniques To Prove If/Then Statements

Let's start by looking at the details of a process that goes on almost automatically in our minds hundreds of times every day—deciding whether something is true or false. Suppose you make the following statement:

If I go home this weekend, I will take my parents out to dinner.

When is your statement true? When is it false; that is, when could you be accused of lying?

The statement we are considering is a composite statement, and its two parts are the following simple statements:

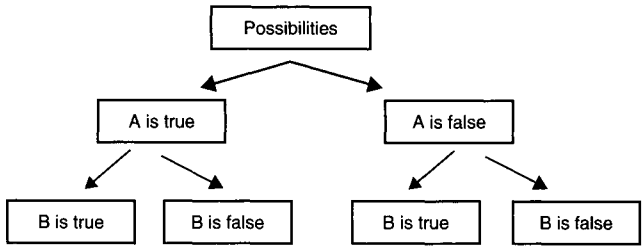
A: I go home this weekend.
B: I will take my parents out to dinner.

As far as your trip is concerned, there are two possibilities:

i. You are going home this weekend (**A** is true).
ii. You are not going home this weekend (**A** is false).

Regarding the dinner, there are two possibilities as well:

i. You will take your parents out to dinner (**B** is true).
ii. You will not take your parents out to dinner (**B** is false).



Thus, we can consider four possibilities:

1. **A** is true and **B** is true.
2. **A** is true and **B** is false.
3. **A** is false and **B** is true.
4. **A** is false and **B** is false.

Case 1. You do go home and you do take your parents out to dinner. Your statement is true.

Case 2. You go home for the weekend, but you do not take your parents out to dinner. You have been caught lying! Your statement is false.

Cases 3 and 4. You cannot be accused of lying if you did not go home, but you did take your parents out to dinner, because they came to visit. If you did not go home, nobody can accuse you of lying if you did not take your parents out to dinner. It is very important to notice that you had not specified what you would do in case you were not going home (**A** is false). So, whether you did take your parents out to dinner or not, you did not lie.

In conclusion, there is only one case in which your statement is false—namely, when **A** is true and **B** is false. This is a general feature of statements of the form "If **A**, then **B**" or "**A** implies **B**."

A statement of the form "If **A**, then **B**" is *true* if we can prove that it is impossible for **A** to be true and **B** to be false at the same time; that is, whenever **A** is true, **B** must be true as well.

The statement "If **A**, then **B**" can be reworded as "**A** is a sufficient condition for **B**" and as "**B** is a necessary condition for **A**." The mathematical use of the words "sufficient" and "necessary" is very similar to their everyday use. If a given statement is true and it provides enough (sufficient) information to reach the conclusion, then the statement is called a *sufficient condition.* If a statement is an inevitable (certain) consequence of a given statement, it is called a *necessary condition.* A condition can be sufficient but not necessary or necessary but not sufficient.

As an example, consider the statement "If an animal is a cow, then it has four legs." Having four legs is a necessary condition for an animal to be a cow, but it is not a sufficient condition for identifying a cow, as it is possible

for an animal to have four legs and not be a cow. On the other hand, being a cow is a sufficient condition for knowing that the animal has four legs. Consult the James & James *Mathematics Dictionary* if you want to find out more about "sufficient" and "necessary" conditions.

All arguments having this form (called *modus ponens*) are valid. The expression "modus ponens" comes from the Latin *ponere*, meaning "to affirm."

Very often one of the so-called *truth tables** is used to remember the information just seen (T = true, F = false):

A	**B**	**If A, then B**
T	T	T
T	F	F
F	T	T
F	F	T

Because in a statement of the form "If **A**, then **B**" the hypothesis and the conclusion are clearly separated (part **A**, the *hypothesis*, contains all the information we are allowed to use; part **B** is the *conclusion* we want to reach, given the previous information), it is useful to try to write in this form any statement to be proved. The following steps can make the statement of a theorem simpler and therefore more manageable, *without changing its meaning:*

1. Identify the hypothesis (**A**) and conclusion (**B**) so the statement can be written in the form "If **A**, then **B**" or "**A** implies **B**."
2. Watch out for irrelevant details.
3. Rewrite the statement to be proved in a form you are comfortable with, even if it is not the most elegant.
4. Check all relevant properties (from what you are supposed to know) of the objects involved.

If you get stuck while constructing the proof, double-check whether you have overlooked some explicit or implicit information you are supposed to know and be able to use in the given context. As mentioned in the General

* Truth tables are diagrams used to analyze composite statements. A column is assigned to each of the simple statements that form the composite statement, then one considers all possible combinations of "true" and "false" for them. The logic connectives used to construct the composite statement (e.g., and, or, if ... then ...) will determine the truth value of the composite statement.

Suggestions section, the proof of a statement depends on the context in which the statement is presented. The examples included in the next sections will illustrate how to use these suggestions, which at this point are somewhat vague, to construct some proofs.

DIRECT PROOF

A direct proof is based on the assumption that the hypothesis contains enough information to allow the construction of a series of logically connected steps leading to the conclusion.

EXAMPLE 1. The sum of two odd numbers is an even number.

Discussion: The statement is not in the standard form "If **A**, then **B**"; therefore, we have to identify the hypothesis and the conclusion. What explicit information do we have? We are dealing with any two odd numbers. What do we want to conclude? We want to prove that their sum is not an odd number. So, we can set:

A. Consider any two odd numbers and add them. (*Implicit hypothesis:* As odd numbers are integer number, we can use the properties and operations of integer numbers.)
B. Their sum is an even number.

Thus, we can rewrite the original statement as: *If we consider two odd numbers and add them, then we obtain a number that is even.* This statement is less elegant than the original one, but it is more explicit because it separates clearly the hypothesis and the conclusion.

From experience we know that the sum of two odd numbers is an even number, but this is not sufficient (good enough) evidence (we could be over-generalizing). We must prove this fact. We will start by introducing some symbols so it will be easier to refer to the numbers used.

Let a and b be two odd numbers. Thus (see the section on facts and properties of numbers at the front of the book), we can write:

$$a = 2t + 1 \text{ and } b = 2s + 1$$

where t and s are integer numbers. Therefore,

$$a + b = (2t + 1) + (2s + 1) = 2t + 2s + 2 = 2(t + s + 1)$$

The number $t + s + 1$ is an integer number, because t and s are integers. This proves that the number $a + b$ is indeed even.

We reached the conclusion that was part of the original statement! We seem to be on the right track. Can we rewrite the proof in a precise and easy to follow way? Let us try!

Proof: Let a and b be two odd numbers. As the numbers are odd, it is possible write:

$$a = 2t + 1 \text{ and } b = 2s + 1$$

where t and s are two integers. Therefore,

$$a + b = (2t + 1) + (2s + 1) = 2t + 2s + 2 = 2(t + s + 1)$$

The number $p = t + s + 1$ is an integer because t and s are integers. Thus,

$$a + b = 2p$$

where p is an integer.

This implies that $a + b$ is an even number. Because this is the conclusion in the original statement, the proof is complete. ■

Let us look back briefly at how this proof relates to the considerations presented at the beginning of this section. We have worked under the assumption that part **A** of the statement is true. We have shown that part **B** holds true, and we have done this using a general way of thinking, not by using specific examples (more about this later). Therefore, it is true that **A** implies **B**. Now, let us consider another statement.

Example 2. If the x- and the y-intercepts of a line that does not pass through the point (0, 0) have rational coordinates, then the slope of the line is a rational number.

Discussion: Let us separate the hypothesis and conclusion:

A. Consider a line in the Cartesian plane such that its x- and y-intercept have rational coordinates, and neither one of them is the point (0, 0).

B. The slope of the line described in the hypothesis is a rational number.

Implicit hypothesis: We need to know the structure of the Cartesian plane, how to find the x- and the y-intercepts of a line, how to find the slope of a line, and how to use the properties and operations of rational numbers.

The hypothesis mentions two special points on the line—namely its x- and y-intercepts. In general, if we know the coordinates of any two points on a line, we can use them to calculate the slope of the line. Indeed, if $A(x_1, y_1)$

and $B(x_2, y_2)$ are any two points on a line (including its x- and y-intercepts), the slope of the line is the number:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

if $x_1 \neq x_2$, and it is undefined if $x_1 = x_2$.

Proof: By hypothesis, if A is the x-intercept of the line, then $A(p/q, 0)$, where $p \neq 0$ (as A is not the point $(0,0)$), $q \neq 0$ (because division by 0 is not defined), and p and q are integers. By hypothesis, if B is the y-intercept of the line, then $B(0, r/s)$, where $r \neq 0$ (as B is not the point $(0,0)$), $s \neq 0$ (because division by 0 is not defined), and r and s are integers. Therefore, the slope of the line is the number:

$$m = \frac{\frac{r}{s} - 0}{0 - \frac{p}{q}} = -\frac{rq}{sp}$$

where $sp \neq 0$, $rq \neq 0$, and sp and rq are both integers. Thus, m, the slope of the line, is a rational number. ■

Example 3. The sum of the first n counting numbers is equal to $[n(n+1)]/2$.

Discussion: We can rewrite this statement in the more explicit (and less elegant) form: "If n is an arbitrary counting number and one considers the sum of the first n counting numbers (*i.e.*, all the numbers from 1 to n, including 1 and n), then their sum can be calculated using the formula $[n(n+1)]/2$.

Let us start by separating the hypothesis and the conclusion:

A. Consider the sum of the first n counting numbers (*i.e.*, $1+2+3+\cdots+n$). (*Implicit hypothesis:* We are familiar with the properties and operations of counting numbers.)

B. The sum above can be calculated using the formula $[n(n+1)]/2$; that is:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Before working on a proof, we might want to check that this equality works for some values of n, but we do need to keep in mind that these will be examples and will not provide a proof.

When $n = 5$, we add the first 5 counting numbers so we have:

$$1+2+3+4+5 = 15.$$

If we use the formula given above, we obtain $5(5+1)/2 = 30/2 = 15$. Clearly, the two answers coincide, but this might be true just by chance. To construct a proof, we need to find a mathematical relationship between the sum of counting numbers and the formula given in the conclusion.

Proof: Let S_n denote the sum of the first n counting numbers; that is:

$$S_n = 1 + 2 + 3 + \cdots + (n-1) + n.$$

Because addition is a commutative operation, we can try rearranging the numbers to write:

$$S_n = n + (n-1) + \cdots + 3 + 2 + 1.$$

Compare these two ways of writing S_n:

$$S_n = 1 + 2 + \cdots + (n-1) + n$$

$$S_n = n + (n-1) + \cdots + 2 + 1.$$

If we add these two equations, we obtain:

$$2S_n = (1+n) + [2 + (n-1)] + \cdots + [(n-1) + 2] + (n+1)$$

or

$$2S_n = (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

or

$$2S_n = n(n+1).$$

From this last equation, we obtain:

$$S_n = \frac{n(n+1)}{2}.$$

Therefore, it is true that $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$ ■

The section on Mathematical Induction includes a different proof of the result presented in Example 3. The proof shown in Example 3 is known as Gauss' proof.

Example 4. If a and b are two positive integers, with $a > b$, then we can find two integers q and r such that:

$$a = qb + r$$

where $0 \leq r < b$ and $0 \leq q$.

Because the statement to be proved is already written in the form "If **A**, then **B**" with the hypothesis and conclusion already separated, we will proceed with the proof.

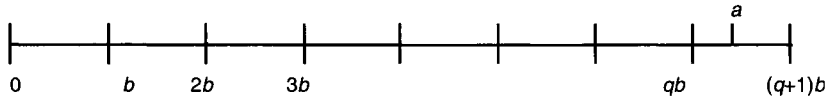
Proof: There are two possibilities: Either a is a multiple of b or a is not a multiple of b. We will consider them separately.

Case 1. If a is a multiple of b, then the statement is proved as $a = qb$, and $r = 0$.

Case 2. We will assume that a is not a multiple of b. This means that if we consider all the multiples of b, none of them will be equal to a. The multiples of b are numbers of the form:

$$b, 2b, 3b, 4b, 5b, \ldots\ldots, nb, (n+1)b, \ldots\ldots$$

This collection is infinitely large, and the values of the numbers get larger and larger. They divide the number line in separate consecutive intervals of size b. As a is a finite number, and these intervals cover the whole positive number line, then a will be in one of the intervals determined by these multiples of b.



Thus,

$$qb < a < (q+1)b$$

for some positive integer q. To show that this number satisfies the conclusion to be reached, we need to show that we can find the other number r. If we subtract qb from these inequalities, we obtain:

$$0 < a - qb < b. \qquad (*)$$

If we now set $r = a - qb$, we can show that this number satisfies the conditions listed in the conclusion.

By the previous inequalities (*), $0 < r < b$. By its definition, $a = qb + r$. Because the two cases presented cover all the possibilities, we proved that the statement is true. ■

The statement in Example 4 is part of the theorem known as the Division Algorithm. Later on we will prove that the numbers q and r we just found are the only ones satisfying the required properties. (See the exercises at the end of the section on Uniqueness Theorems.) To be accurate, Example 4 could have been included in the section on Existence Theorems, because it states that there exist two numbers having certain properties. Moreover, its

proof is constructed by considering two separate cases. Thus, one could argue for its inclusion in the section on Multiple Hypotheses!

Example 5. A five-digit number is divisible by 3 when the sum of its digits is divisible by 3.

Discussion: This statement can be rewritten as: If the sum of the digits of a five digit number is divisible by 3, then the number is divisible by 3.

A. Let n be an integer number with $n = \pm a_4a_3a_2a_1a_0$, $0 \leq a_i \leq 9$ for all $i = 0, 1, 2, 3, 4$, and $a_4 \neq 0$, such that $a_4 + a_3 + a_2 + a_1 + a_0 = 3t$, where t is an integer number. (The fact that n is an integer number is an implicit hypothesis, as the concept of divisibility is defined only for integer numbers. The $\pm$ sign indicates that the number n can be either positive or negative.)

B. The number n is divisible by 3; that is, $n = 3s$, where s is an integer number.

Proof: As the hypothesis provides information about the digits of the number, we will separate the digits using powers of 10. For the sake of simplicity, let us assume that n is positive. Thus,

$$n = a_4a_3a_2a_1a_0 = 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0.$$

By hypothesis, $a_4 + a_3 + a_2 + a_1 + a_0 = 3t$, where t is an integer number. Therefore,

$$a_0 = 3t - a_4 - a_3 - a_2 - a_1$$

If we substitute this expression for a_0 into the expression for n and perform some algebraic steps, we obtain:

$$\begin{aligned} n &= 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \\ &= 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + (3t - a_4 - a_3 - a_2 - a_1) \\ &= 9{,}999a_4 + 999a_3 + 99a_2 + 9a_1 + 3t. \end{aligned}$$

Therefore,

$$\begin{aligned} n &= 9{,}999a_4 + 999a_3 + 99a_2 + 9a_1 + 3t \\ &= 3(3{,}333a_4 + 333a_3 + 33a_2 + 3a_1 + t). \end{aligned}$$

Because the number $3{,}333a_4 + 333a_3 + 33a_2 + 3a_1 + t$ is an integer, we proved that number n is divisible by 3.

If n is a negative number, just replicate all the steps above, starting with $n = -a_4a_3a_2a_1a_0$. ■

Note: The number of digits used in Example 5 is irrelevant. This is a special case of the much more general statement: "A number is divisible by 3 when the sum of its digits is divisible by 3." We chose to use five digits because the proof of the more general statement, which at the beginning is very similar to the one above, can be easily completed using a technique that will be introduced later—namely, "proof by induction." Let's look at the setup of the general proof.

Let n be an integer number with $n = a_k a_{k-1} \ldots a_2 a_1 a_0$, $0 \leq a_i \leq 9$ for all $i = 0, 1, 2, \ldots, k$ and $a_k \neq 0$, such that $a_k + a_{k-1} + \cdots + a_2 + a_1 + a_0 = 3t$, where t is an integer number. Then, following the same steps performed in the proof in Example 6, we can write:

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10^2 a_2 + 10a_1 + a_0 \\ &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10^2 a_2 + 10a_1 + (3t - a_k - a_{k-1} - \cdots - a_2 - a_1) \\ &= (10^k - 1)a_k + (10^{k-1} - 1)a_{k-1} + \cdots + 99a_2 + 9a_1 + 3t. \end{aligned}$$

At this point, to be able to show that n is divisible by 3, we need to prove that $10^s - 1$ is divisible by 3 for all $s \geq 1$. This is the step that can require proof by induction (see Exercise 8 at the end of the section on Mathematical Induction), unless one is familiar with modular arithmetic. As already mentioned, as one's mathematical background increases, one has more tools to use and therefore becomes able to construct the proof of a statement using several different approaches.

Example 6. Let f and g be two real-valued functions defined for all real numbers and such that $f \circ g$ is well defined for all real numbers. If both functions are one-to-one, then $f \circ g$ is a one-to-one function.

Discussion: We will separate the hypothesis and the conclusion:

A. We are considering two functions that have the following properties:

1. They are defined for all real numbers.
2. They are one-to-one.

The fact that the functions are called f and g is irrelevant. We can use any two symbols, but having a quick way to refer to the functions does simplify matters.

B: The function is one-to-one.

We can fully understand the meaning of the given statement only if we are familiar with the definitions of function, one-to-one function, and composition of functions. (See the section on facts and properties of functions at the front of the book.)

By definition, a real-valued function h is said to be *one-to-one* if for every two real numbers x_1 and x_2 such that $x_1 \neq x_2$, it follows that $h(x_1) \neq h(x_2)$.

The composition of two functions is the function defined as $f \circ g(x) = f(g(x))$.

Using these two definitions, we can explicitly rewrite the conclusion to be reached as:

B. If x_1 and x_2 are two real numbers such that $x_1 \neq x_2$, then

$$f \circ g(x_1) \neq f \circ g(x_2).$$

Proof: Let x_1 and x_2 be two real numbers such that $x_1 \neq x_2$. (Examine in detail each step of the construction of the function to see "what happens" to the values corresponding to x_1 and x_2.) As g is a one-to-one function, it follows that:

$$g(x_1) \neq g(x_2).$$

Set $y_1 = g(x_1)$ and $y_2 = g(x_2)$.

As f is a one-to-one function and $y_1 \neq y_2$, it follows that:

$$f(y_1) \neq f(y_2)$$

that is, $f(g(x_1)) \neq f(g(x_2))$.

Thus, by the definition of, we can conclude that if x_1 and x_2 are two real numbers such that $x_1 \neq x_2$, then

Therefore is $f \circ g$ a one-to-one function. ■

RELATED STATEMENTS

If we are having a conversation with a person who makes a statement whose meaning escapes us, we might ask, "What do you mean?" Our hope is that the wording of the statement will be changed so that we can grasp its meaning, without changing the meaning itself. This situation happens when we are working with mathematical statements as well. How do we change a mathematical statement into one that is easier for us to handle but has the same mathematical meaning?

Two statements are *logically equivalent* when they have the same truth table. So, we can change a mathematical statement into one easier to work

with and which is true or false exactly when the original statement is true or false.

Given the statement **A**, we can construct the statement "not **A**," which is false when **A** is true and true when **A** is false; "not **A**" is the *negation* of **A**. Clearly, these two statements are related, but they are not logically equivalent.

As most mathematical statements are in the form "If **A**, then **B**," we will work in detail on the three statements related to "If **A**, then **B**," and defined as follows:

- The *converse* of the statement "If **A**, then **B**" is the statement "If **B**, then **A**." (To obtain the converse, reverse the roles of the hypothesis and the conclusion.)
- The *inverse* of the statement "If **A**, then **B**" is the statement "If 'not **A**,' then 'not **B**.'" (To obtain the inverse, negate the hypothesis and the conclusion.)
- The *contrapositive* of the statement "If **A**, then **B**" is the statement "If 'not **B**,' then 'not **A**.'" (To obtain the contrapositive, construct the converse, and then consider the inverse of the converse. This means reverse the roles of the hypothesis and the conclusion and negate them.)

Let us consider an example to clarify these definitions. Let the original statement be:

If x is a rational number, then x^2 is a rational number.

Its *converse* is the statement "If x_2 is a rational number, then x is a rational number."

Its *inverse* is the statement "If x is not a rational number, then x_2 is not a rational number."

Its *contrapositive* is the statement "If x_2 is not a rational number, then x is not a rational number."

These statements cannot be all logically equivalent because the original statement and its contrapositive are true (prove that this claim is indeed correct), while the converse and the inverse are false (why?). To avoid guessing if and when these four statements are logically equivalent, we will construct their truth tables. Because we want to compare the four tables, the columns for the statements **A** and **B** must always be the same. We want to know when, under the same conditions for **A** and **B**, we get the same conclusions regarding the truth of the final composite statements.

Following is the truth table for the statement "If **A**, then **B**":

A	**B**	**If A, then B**
T	T	T
T	F	F
F	T	T
F	F	T

Following is the truth table for the *converse* statement "If **B**, then **A**.":

B	**A**	**If B, then A**
T	T	T
F	T	T
T	F	F
F	F	T

Following is the truth table for the *inverse* statement "If 'not **A**,' then 'not **B**' ":

A	**B**	**Not A**	**Not B**	**If 'not A,' then 'not B'**
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Following is the truth table for the *contrapositive* statement "If 'not **B**,' then 'not **A**' ":

A	**B**	**Not B**	**Not A**	**If 'not B,' then 'not A'**
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

We can conclude from the preceding tables that the original statement is only equivalent to its contrapositive. The converse and the inverse are logically equivalent to each other, but not to the original statement.

PROOF BY CONTRAPOSITIVE (AKA PROOF BY CONTRADICTION OR INDIRECT PROOF)

In some cases we cannot use the kind of direct, straightforward arguments we have already seen; that is, we cannot deduce conclusion **B** directly from hypothesis **A**. This might happen because assuming that **A** is true does not seem to give us enough information to allow us to prove that **B** is true. In other cases, direct verification of the conclusion **B** would be too time consuming or impossible. Therefore, we must find another starting point.

Because a statement is logically equivalent to its contrapositive, we can try to work with the contrapositive. This gives us a different starting point because we will start by assuming that **B** is false, and we will prove that this implies that **A** is false, as the contrapositive of the original statement is "If 'not **B**,' then 'not **A**.' "

Let us consider an example that illustrates the use of this technique.

EXAMPLE 7. Let n be an integer number. If the number $7n+4$ is even, then n is even.

Discussion: In this case, the hypothesis and the conclusion are clearly distinguishable; therefore we can set:

A. The number $7n+4$ is even. (*Implicit hypothesis:* We can use the properties of integer numbers and their operations.)
B. The number n is even.

By hypothesis, $7n+4=2k$ for some integer number k (see the section on facts and properties of numbers at the front of the book). If we try to solve for n explicitly, we will need to divide by 7, and it is not evident that the result of the division will be an integer and will give information on the parity of n. Therefore, we will try to prove the original statement by using its contrapositive.

Proof: Assume that **B** is false, and that "not **B**" is true, and use this as the new hypothesis. We will start by assuming that "n is not even." This means that "n is odd." Thus,

$$n = 2t + 1$$

for some integer number t. Using this information to calculate the number $7n+4$ yields:

$$7n+4 = 7(2t+1)+4 = 2(7t+5)+1.$$

The number $s = 7t+5$ is an integer because 7, t, and 5 are integers; therefore, we can write:

$$7n+4 = 2s+1$$

and conclude that $7n+4$ is an odd number ("not **A**"). This means that the statement "If 'not **B**,' then 'not **A**' " is true, and its contrapositive (*i.e.*, the original statement) is also true. ■

Example 8. Let m and n be two integers. If they are consecutive, then their sum, $m+n$, is an odd number.

Discussion: In this case we can set:

A. The numbers m and n are consecutive
(*i.e.*, if m is the larger of the two, then $m-n=1$).
(*Implicit hypothesis:* We can use the properties of integer numbers and their operations.)

B. The sum $m+n$ is an odd number.

This statement can be proved either directly (this proof is left as an exercise) or by using its contrapositive.

Proof: Assume that **B** is false and "not **B**" is true, and use this as the new hypothesis. We will start by assuming that the number $m+n$ is not odd. Then, $m+n$ is even and there exists some integer number k such that $m+n=2k$. This implies that $m=2k-n$, thus:

$$m-n = (2k-n)-n = 2(k-n).$$

As the number $k-n$ is an integer, we have proved that the difference between m and n is an even number; therefore, it cannot equal 1, and m and n are not consecutive numbers. This means that the statement "If 'not **B**,' then 'not **A**' " is true, and its contrapositive (*i.e.*, the original statement) is also true. ■

Example 9. There are infinitely many prime numbers.

Discussion: We need to analyze the statement to find the hypothesis and the conclusion because they are not clearly distinguishable. The point is that we

want to consider the collection of all prime numbers and show that it is infinite. Therefore, we can set:

A. Consider the collection of all prime numbers, and call it P. (*Implicit hypothesis:* We can use the properties of prime numbers and the operations and properties of counting numbers, because prime numbers are counting numbers.)
B. The collection P is infinite.

To prove this statement directly, we have to show that we "never run out of prime numbers." Direct proof is not a feasible method, because if the conclusion is true, it would take an infinite amount of time to list all prime numbers. Even if the statement is false, there could be millions of prime numbers, and listing them would take a long time. Moreover it is not always easy to decide if a large number is prime, even by using a computer.

Proof: We will assume that the conclusion we want to reach is false. So we will start by assuming that the collection P of prime number is indeed finite. Then we can have a complete list of prime numbers:

$$p_1 = 2;\ p_2 = 3;\ p_3 = 5;\ p_4 = 7; \ldots; p_n.$$

Using the trichotomy property of numbers, we can list the prime numbers in increasing order. So,

$$p_1 < p_2 < p_3 < p_4 < \cdots < p_n.$$

Thus, p_n is the largest existing prime number.
(Let's have a little more discussion at this point, because we could ask the question, "Where do we go now?" We could try to construct a prime number that is not in the list using the ones in the list. How do we reach this goal? We can try to use operations with counting numbers. Division is not an easy one to use, because the quotient of two integers might be a non-integer number. If we consider the sum of all the prime numbers listed above, we do not have a lot of information. We do not know how many integers we are using, so we do not even know if the sum is an even or odd number. We can consider the product of all these primes. We have a little more information about this product. It is not a prime number, because it is divisible by all the prime numbers. We know that it is even because 2 is one of the prime numbers in the list. All prime numbers larger than 2 are odd. So, what can we do? We can always add 1 to an even number to construct an odd number—that is, its consecutive.)

Consider the number $q = p_1p_2p_3p_4 \ldots p_n + 1$. The number q is odd and it is larger than all the prime numbers listed, so it is not one of the

numbers listed. Moreover, q is not divisible by any of the prime numbers because the quotient:

$$\frac{q}{p_k} = p_1 p_2 \cdots p_{k-1} p_{k+1} \cdots p_n + \frac{1}{p_k}$$

is not an integer. This implies that q is a prime number because it is not divisible by any prime number. But, we had assumed that we had a complete list of prime numbers; therefore, the collection P of all prime numbers is infinite. ■

HOW TO CONSTRUCT THE NEGATION OF A STATEMENT

The truth of some statements can be proved in more than one way, either by using direct proof or using proof by contrapositive. Generally, if we have a choice, we should use direct proof. Indeed, direct proofs are usually more intuitively understood and more informative. Moreover, when using proof by contrapositive, there is an important point that needs to be addressed. We have to construct the statement "not **B**" to use as the hypothesis, and this can be a tricky step. Sometimes it is enough to insert the word *not* in **B** to achieve our goal, as it happens in the previous examples. The statements "$x+y$ is irrational" and "the collection is infinite" are changed into the statements "$x+y$ is not irrational" and "the collection is not infinite."

Other cases are not so easy to handle, especially when **B** includes words such as *unique*, *for one*, *for all*, *every*, and *none*. These expressions are usually called *quantifiers*. Let us see how we can work with some of these expressions.

Original Statement	**Negative**
At least one	None
Some	None
All objects in a collection have a certain property	There is at least one object in the collection that does not have that property
Every object in a collection has a certain property	There is at least one object in the collection that does not have that property
None	There is at least one
There is no	There is at least one

Quantifiers are not the only possible source of problems when constructing the negation of a statement. The logical connectors "or" and "and" have to be handled carefully as well.

The composite statement "**C** *or* **D**" is true when either one of the statements **C** or **D** is true. While it is possible for both statements to be true, it is not required. Unless otherwise indicated, the "or" used in mathematics is inclusive; that is, it includes the possibility that both parts of the statement are true. This use of "or" is different from its everyday use, when "or" suggests a choice between two possibilities (as in, "Would you like to have coffee or tea?"); therefore, for the statement "**C** *or* **D**" to become false, both **C** and **D** must be false. Thus, the negation of "**C** *or* **D**" (*i.e.*, the statement "not '**C** *or* **D**' ") is the statement " 'not **C**' *and* 'not **D**.' "

The composite statement "**C** *and* **D**" is true when both statements **C** and **D** are true. Therefore, for it to become false, it is sufficient that either **C** or **D** is false; thus, the negation of "**C** *and* **D**" (*i.e.*, the statement "not '**C** *and* **D**' ") is the statement " 'not **C**' *or* 'not **D**.' "

The truth tables can reinforce and clarify what has just been stated in the previous paragraphs:

Following is the truth table for the *negative* of the statement "**C** or **D**" (*i.e.*, "not '**C** or **D**' ").

C	**D**	**C or D**	**Not "C or D"**
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Following is the truth table for the statement " 'not **C**' and 'not **D**.' "

C	**D**	**Not C**	**Not D**	**"Not C" and "Not D"**
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

By comparing these two tables, we can conclude that the statements "not '**C** *or* **D**' " and " 'not **C**' *and* 'not **D**' " are indeed logically equivalent. The constructions of the truth tables for the statements "not '**C** *and* **D**' " and " 'not **C**' *or* 'not **D**' " is left as an exercise.

Again, the method of using the contrapositive of a statement should be used when the assumption that **A** is true does not give a good starting point, but the assumption that **B** is false does. Sometimes the statement whose truth we are trying to establish gives us a hint that it might be easier to work with its contrapositive. This method is helpful if **B** already contains a "not," because if we negate **B** we get an affirmative statement.

Example 10. The graphs of the functions $f(x) = -\frac{x}{6} + \frac{1}{4}$ and $g(x) = \frac{x-1}{x^2+x-2}$ have no points in common.

Discussion: If we try to separate hypothesis and conclusions, we obtain:

A. Consider the graphs of the functions defined above. (*Implicit hypothesis:* We are familiar with the concepts of functions, graphs, and intersection points, as well as real numbers and their operations and properties.)

B. The graphs of the two functions do not have any point in common.

We could construct the graphs of both functions. The graph of f is a line, so it is easy to obtain. The graph of g is more complicated. This function is not defined at $x = -2$ and at $x = 1$. Thus, its graph has three parts: one for $x < -2$, one for x in the interval $(-2, 1)$, and one for $x > 1$. Moreover, some mathematicians object to the use of graphs as proofs, arguing that it is possible for the graphs of the functions not to have points in common in the finite part we graphed but to have points in common in the parts we have not graphed. So, they expect a proof that supports what we can observe by graphing. Therefore, it might just be easier to prove the truth of the contrapositive of the original statement.

We have to deny the statement: "The graphs of the two functions do not have any point in common," and this is easy to do because the statement is already negative. So we obtain: "The graphs of the two functions have at least one point in common."

Proof: We start by assuming that the graphs of the two functions have a point in common. This means that the equation:

$$f(x) = g(x)$$

has at least one real solution.

As the function g is not defined for $x = -2$ and $x = 1$, we can assume that $x \neq -2$ and $x \neq 1$ in the equation:

$$-\frac{x}{6} + \frac{1}{4} = \frac{x-1}{x^2+x-2}.$$

If we factor the denominator in the right-hand side, we obtain:

$$-\frac{x}{6}+\frac{1}{4}=\frac{x-1}{(x-1)(x+2)}.$$

We can divide by $x-1$ because $x\neq 1$, so $x-1\neq 0$, and the equation becomes:

$$-\frac{x}{6}+\frac{1}{4}=\frac{1}{x+2}.$$

We can multiply by $x+2$ because $x\neq -2$, so $x+2\neq 0$ to obtain:

$$2x^2+x+6=0.$$

By hypothesis, this equation has at least one real solution, contradicting the fact that a quadratic equation with a negative discriminant ($\Delta = 1-4(6)(2)=-47$) has no real solutions (implicit hypothesis). Therefore, the two graphs have no points in common. ■

Example 11. Every positive number smaller than 1 is larger than its square.

Discussion:

A. Consider the collection of all positive real numbers smaller than 1. (*Implicit hypothesis:* We are familiar with all properties and operations of real numbers.)

B. Every number in the collection described in **A** is larger than its square.

We will prove this statement in two ways and compare the proofs.

1. Direct Proof

Let x be a positive number smaller than 1; that is, $x < 1$. We can multiply both sides of this inequality by x to obtain:

$$x^2 < x.$$

Therefore, x is larger than its square.

2. Proof by Contrapositive

We need to construct the statement "not **B**." For "not **B**" there is at least one positive real number smaller than 1 that is smaller than its square. Thus,

we can start from the assumption that there exists a positive number x such that:

$$x < x^2.$$

We can rewrite the previous inequality as:

$$x - x^2 < 0.$$

The expression on the left-hand side of the inequality can be factored, and the inequality becomes:

$$x(1 - x) < 0.$$

The product of two real numbers is negative if and only if the numbers have opposite sign. Because, by hypothesis, $x > 0$, we must conclude that:

$$1 - x < 0; \quad \text{that is, } 1 < x.$$

Thus, we proved that "not **A**" is true. Because the contrapositive of the original statement is true, the original statement is true as well. ■

In Example 11, the direct proof is shorter and simpler, but sometimes it is nice to know that there is another way to achieve a goal.

As already mentioned, one has to use caution identifying the hypothesis and conclusion to construct and use the contrapositive of a statement. Sometimes a theorem might include "overarching" hypotheses that are just used to define the general setting in which the statement "if **A**, then **B**" should be considered. This kind of hypothesis will not be changed. For example, consider the following statement: "Let x, y, and z be counting numbers. If xy is not a multiple of z, then x is not a multiple of z and y is not a multiple of z." When we construct its contrapositive, we do not deny the fact that x, y, and z are counting numbers. We will instead consider the statement: "Let x, y, and z be counting numbers. If x is a multiple of z or y is a multiple of z, then xy is a multiple of z."

Denying a statement that contains more than one quantifier can be difficult at first. Some of these statements are found in calculus and real analysis, in the definitions of limits and continuity. Let's examine one of them:

A. The real number L is said to be the limit of the function $f(x)$ at the point c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

This statement specifies that for L to be the limit every $\varepsilon > 0$ must have a certain property. Therefore, to construct "not **A**," one must require that there is at least one $\varepsilon > 0$ without that property. Thus,

"Not A". The real number L is not the limit of the function $f(x)$ at the point c if there exists at least one $\varepsilon > 0$ such that for all $\delta > 0$ there exists an x with $0|x - c| < \delta$ and $|f(x) - L| > \varepsilon$.

For more details on the definition of limits, and proofs regarding them, see the section on Limits.

EXERCISES

Given the following statements, negate them:

1. The function f is defined for all real numbers.
2. Let x and y be two numbers. There is a rational number z such that $x + z = y$.
3. The function f has the property that for any two distinct real numbers x and y, $f(x) \neq f(y)$.
4. The equation $P(x) = 0$ has only one solution. (Assume it is known that the equation has at least one solution.)
5. All nonzero real numbers have nonzero opposites.
6. For every number $n > 0$, there is a corresponding number $M_n > 0$ such that $f(x) > n$ for all real numbers x with $x > M_n$. (To understand this statement better, you might want to use a graph in the Cartesian plane.)
7. Every number satisfying the equation $P(x) = Q(x)$ is such that $|x| < 5$.
8. The equation $P(x) = 0$ has only one solution. (Check Exercise 4.)
9. The function f is continuous at the point c if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.
10. For every real number x the number $f(x)$ is rational.

Given the following statements, construct their (a) contrapositive, (b) converse, and (c) inverses.

11. If x is an integer divisible by 6, then x is divisible by 2.
12. If a quadrilateral is not a parallelogram, then its diagonals do not bisect.
13. If the two polynomials:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \text{ and}$$

$$Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

are equal for all real numbers x, then $a_i = b_i$, for all i, with $0 \leq i \leq n$.

14. If two integer numbers are odd, their product is odd.
15. If the product of two integer numbers is even, then at least one of the numbers is even.

Using the techniques seen in this section, prove the following statements:

16. Let f and g be two nondecreasing functions such that $f \circ g$ is well defined. Then $f \circ g$ is nondecreasing. (See front material on functions for the definition of nondecreasing.)
17. If x is a rational nonzero number and y is an irrational number, then the number xy is irrational.
18. Let n be a number with three or more digits. If the two-digit number made by n's two rightmost digits is divisible by 4, then n is divisible by 4.
19. If $(a+b)^2 = a^2 + b^2$ for all real numbers b, then a must be zero.
20. Let n be a counting number. If the number $2^n - 1$ is a prime number, then n is prime.
21. Every four-digit palindrome number is divisible by 11. (A palindrome number reads the same forward or backward.)
22. Let f be a nondecreasing function defined for all real numbers. Then, for all $x \neq c$:
$$\frac{f(c) - f(x)}{c - x} \geq 0.$$
(See front material on functions for the definition of nondecreasing.)
23. Prove that the following statement is true in two ways, directly and by using the contrapositive method:
The function $f(x) = mx + b$, with $m \neq 0$, is a one-to-one function.
(See front material on functions for the definition of one-to-one function.)
24. Let f and g be two real valued functions defined for all real numbers and such that $f \circ g$ is well defined. If f and g are onto, so is $f \circ g$. (See front material on functions for the definition of onto function.)

Read the following proofs and make sure you understand them. Then, outline the proofs, listing explicitly all the most important steps. Fill in details that might have been skipped (*e.g.*, write the statement in the form "If A, then B," indicate which technique has been used, fill in logic details and algebraic steps.)

25. Euclid's algorithm for finding the greatest common divisor of two numbers:
Let a and b be two positive integers with $a > b$. Divide a by b and write:
$$a = bq_1 + r_1$$

with $q_1 \geq 0$ and $0 \leq r_1 < b$.

Then divide b by r_1, obtaining:

$$b = r_1 q_2 + r_2$$

with $q_2 \geq 0$ and $0 \leq r_2 < r_1$.
Continuing, we can divide r_1 by r_2 to obtain:

$$r_1 = r_2 q_3 + r_3$$

with $q_3 \geq 0$ and $0 \leq r_3 < r_2$.
Continue this process as long as $r_1 \neq 0$.
Then, the greatest common divisor of a and b, denoted as (a,b) or $GCD(a,b)$ is the last nonzero remainder. (See front of the book for the definition of the greatest common divisor of two numbers.)

Proof: If we use the process described in the statement, we obtain:

$$\begin{aligned}
a &= bq_1 + r_1 \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
r_{n-3} &= r_{n-2} q_{n-1} + r_{n-1} \\
r_{n-2} &= r_{n-1}\, q_n + r_n \\
r_{n-1} &= r_n\, q_{n+1} + 0.
\end{aligned}$$

The process will take at most b steps because $b > r_1 > r_2 > \ldots \geq 0$.
The last of the equalities written above implies that $r_n = GCD(r_{n-1}, r_n)$. (Explain why.)
Because

$$\begin{aligned}
r_{n-2} &= r_{n-1}\, q_n + r_n \\
&= r_n\, q_{n+1}\, q_n + r_n = r_n\, t_1
\end{aligned}$$

with $t_1 > 0$, it follows that r_n divides r_{n-2} and r_{n-1}. So, r_n is a common divisor of r_{n-2} and r_{n-1}.
If d is another positive integer divisor of r_{n-2} and r_{n-1}, then d will divide r_n. (Check this claim.) Therefore,

$$r_n = GCD(r_{n-1}, r_{n-2}).$$

Similarly, working backward through all the equalities of the algorithm, we obtain that $r_n = GCD(r_{n-2}, r_{n-3}), \ldots, r_n = GCD(a, b)$. ■

26. If $d = GCD(a, b)$, then $d = sa + tb$ for some integers s and t.
Proof: Using the steps of the Euclidean algorithm described in Exercise 25, we obtain:

$$r_1 = a - bq_1$$

$$r_2 = b - r_1 q_2 = b - (a - bq_1)q_2 = as_2 + bt_2$$

$$r_3 = r_1 - r_2 q_3 = (a - bq_1) - (as_2 + bt_2)q_3 = as_3 + bt_3.$$

Proceeding in this way, in at most b steps we will be able to write:

$$r_n = sa + tb.$$

The statement is therefore proved. ■

27. Let p be a prime number. If p divides the product ab, then p divides either a or b.

Proof: If p does not divide a, then $GCD(a, p) = 1$. (Explain why.) Therefore,

$$1 = sa + pt$$

for some integers s and p. (See Exercise 26.)
Thus,

$$\begin{aligned} b &= b(sa + pt) \\ &= (kp)s + bpt \\ &= p(ks + bt). \end{aligned}$$

This implies that p divides b. (Explain why.) ■

28. Let p be a prime number. Then, $\sqrt{p}$ is an irrational number.
Proof: Let us assume that $\sqrt{p}$ is a rational number; that is,

$$\sqrt{p} = \frac{n}{q}$$

where $n \neq 0$, $q \neq 0$, and n and q are integers, with the fraction written in reduced form. (See front material on rational numbers.) Therefore,

$$p = \frac{n^2}{q^2}.$$

Thus,

$$n^2 = pq^2.$$

Because n^2 is a multiple of p, which is a prime number, then n must be a multiple of p. (See Exercise 27.) Therefore, we can write $n = pk$ for some positive integer k. This implies:

$$p^2k^2 = pq^2$$

or

$$pk^2 = q^2.$$

Because q^2 is a multiple of p, which is a prime number, then q must be a multiple of p. (See Exercise 27 and explain how to use it in this case.) Therefore, we can write $q = pm$ for some positive integer m and

$$\frac{n}{q} = \frac{pk}{pm} = \frac{k}{m}.$$

This contradicts the fact that the fraction n/q is already in reduced form, and proves that $\sqrt{p}$ is an irrational number. ■

Special Kinds of Theorems

There are certain kinds of theorems whose proofs follow rather standard structures. In this chapter, we will look at some of the most important and common of these special theorems.

"IF AND ONLY IF" OR "EQUIVALENCE THEOREMS"

Statements including the expression "if and only if" are rather common and very useful in mathematics. If we can show that "**A** if and only if **B**," we are proving that **A** and **B** are (logically) equivalent statements, because either one of them is true (or false) only when the other one is true (or false). The statement "**A** if and only if **B**" means that "**A** is a necessary and sufficient condition for **B**" and that at the same time "**B** is a necessary and sufficient condition for **A**."

Thus, to prove that the statement "**A** if and only if **B**" is true, we must prove that:

1. If **A**, then **B**. (**A** is a sufficient condition for **B**; **B** is a necessary condition for **A**.)
2. If **B**, then **A**. (**B** is a sufficient condition for **A**; **A** is a necessary condition for **B**.)

Therefore, the proof of an "if and only if" statement has two parts. We can use any one of the techniques we know to construct each part.

Notice that the statements "If **A**, then **B**" and "If **B**, then **A**" are converses of each other.

Example 1. A nonzero real number is positive if and only if its reciprocal is positive.

Proof: Consider:

A. A real number a is positive.
B. The reciprocal of a, denoted as a^{-1}, is positive.

Part 1. If **A**, then **B**.
The fact that the number a is positive is sufficient to imply that its reciprocal is positive. By definition of a reciprocal:

$$a \times a^{-1} = 1.$$

So, the number $a \times a^{-1}$ is positive.

By the properties of operations of real numbers, the product of two numbers is positive only if the two numbers are either both positive or both negative. Because by hypothesis a is positive, it follows that a^{-1} is positive.

Part 2. If **B**, then **A.**
The fact that the number a is positive is necessary to imply that its reciprocal is positive. By definition of reciprocal

$$a \times a^{-1} = 1.$$

So, the number $a \times a^{-1}$ is positive.

By the properties of operations of real numbers, the product of two numbers is positive only if the two numbers are either both positive or both negative. By hypothesis, a^{-1} is positive, thus it follows that a is positive. ■

It is easy to see that the two parts of the proof in Example 1 are very similar. Thus, after making sure that no details have been overlooked, we can edit and streamline the proof. The following is an example of how the proof can be condensed.

Let $a \times a^{-1} = 1$. The product of two numbers is positive if and only if the two numbers are either both positive or both negative. Thus, if a is positive, so is a^{-1}, and, conversely, if a^{-1} is positive, so is a.

Example 2. A counting number is odd if and only if its square is odd.

Proof: Let n represent a generic counting number. Then we can set:

A. The number n is odd.
B. The number n^2 is odd.

Part 1. If **A**, then **B**.
The fact that the number n is odd is sufficient to imply that its square is odd. By hypothesis the number n is odd. So we can write $n = 2p + 1$, where p is an integer number. Therefore,

$$\begin{aligned} n^2 &= (2p+1)^2 \\ &= 4p^2 + 4p + 1 \\ &= 2(2p^2 + 2p) + 1. \end{aligned}$$

Because p is an integer number, the number $s = 2p^2 + 2p$ is integer as well. Thus,

$$n^2 = 2s + 1.$$

This proves that n^2 is odd.

Part 2. If **B**, then **A.**
The fact that the square of number n is odd is sufficient to imply that the number itself is odd.

Discussion: If we know that n^2 is odd, we can only write $n^2 = 2t + 1$, with t positive integer. We cannot write that $n^2 = (2k + 1)^2$, because this is the conclusion we are trying to reach. If $n^2 = 2t + 1$, then

$$n = \sqrt{n^2} = \sqrt{2t + 1}.$$

This equality does not give us any useful information. So, we need to look for another starting point. We can try to prove its contrapositive. Let us assume "not **A**"; that is, the number n is not odd.

Because n is an even number, it can be written as $n = 2t$, with t positive integer; therefore, $n^2 = 4t^2$. This implies that n^2 is even, as we can write it as $n^2 = 2(2t^2)$, and $2t^2$ is an integer number.

Thus, we have proved that "not **A**" implies "not **B**." So, the statement "If **B**, then **A**" is true. ■

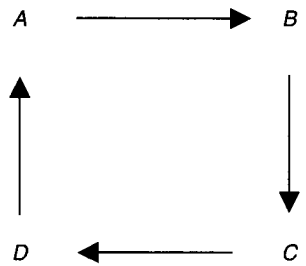
As can be seen from Example 2, the proof of an equivalence theorem might require the use of different techniques (e.g., direct proof and use of the contrapositive) for the different parts of the proof.

Some theorems list more than two statements and claim that they are all equivalent. The construction of the proof of these theorems is rather flexible (that is, it can be set up in several ways), as long as we establish that each statement implies each of the other statements and that each statement is implied by each of the other statements. In this way we prove that each statement is sufficient and necessary for all the others. Some of the

implications will have to be proved explicitly, while others might follow from some of the implications already proved.

Let us assume that we want to prove that four statements, **A**, **B**, **C**, and **D**, are equivalent. There are many ways of proceeding. We will look at four of them, working in detail on the first one and just giving the outlines (diagrams) for the others. It will be up to you to check that by proving the implications represented by the arrows in each diagram, we would indeed prove that the four statements **A**, **B**, **C**, and **D** are equivalent.

Diagram 1



We use this diagram to express the fact that we have proved the following implications:

i. If **A**, then **B**.
ii. If **B**, then **C**.
iii. If **C**, then **D**.
iv. If **D**, then **A**.

The order in which the implications have been proved is not relevant.

We can see that, if these four implications are true, then **A** implies **B**, **C**, and **D**. Indeed:

a. **A** implies **B** (proved explicitly).
b. **A** implies **B**, and **B** implies **C**; so **A** implies **C**.
c. **A** implies **C**, and **C** implies **D**; so **A** implies **D**.

On the other hand, **A** is implied by the other three statements:

a. **D** implies **A** (proved explicitly).
b. **C** implies **D**, and **D** implies **A**; so **C** implies **A**.
c. **B** implies **C**, and **C** implies **A**; so **B** implies **A**.

Similarly, we can establish that **B**, **C**, and **D** imply all other statements and are implied by all of them.

Diagram 2

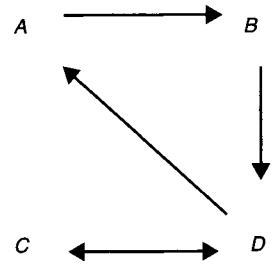


Diagram 3

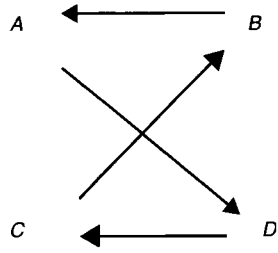
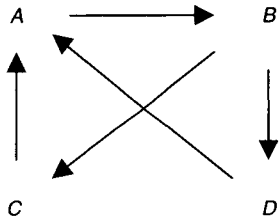


Diagram 4



Depending on our priorities, we can choose the chain of proofs that involves the implications that are easier to prove, or the one that gives the more detailed information, or the one that requires the smallest number of proofs. Therefore, there is no prescribed way of proving that three or more statements are equivalent.

In general, to prove that n statements are equivalent, one needs to prove at least n implications (see Diagram 1 for an illustration of this claim).

Let us now consider some more examples.

Example 3. Let a and b be two distinct real numbers. Then the following statements are equivalent:

i. The number b is larger than the number a.
ii. Their average, $(a + b)/2$, is larger than a.
iii. Their average, $(a + b)/2$, is smaller than b.

Discussion: By hypothesis a and b are two distinct real numbers, so we can use all the properties of real numbers, such as the order (or trichotomy) property, because the statements deal with comparison of numbers. The order property states that, given any two real numbers, x and y, one of the three following relations holds: $x < y$, or $x > y$, or $x = y$.

To prove that the three statements are equivalent we will need at least three separate proofs. We will construct four proofs, according to the following diagram:

$$\begin{array}{ccc} i & \leftrightarrow & ii \\ \updownarrow & & \\ iii & & \end{array}$$

Proof:

Part 1. If i, then ii; that is:

> If the number b is larger than the number a,
> then their average is larger than a.

By hypothesis:

$$a < b.$$

Because we want to obtain $a + b$, we can add either a or b to both sides of the inequality. Because the conclusion we want to reach deals with a, we could try adding a. Thus, we obtain:

$$2a < a + b.$$

Dividing by 2 yields:

$$a < \frac{a+b}{2}.$$

This proves that the conclusion holds true. So, the statement "If i, then ii" is true.

Part 2. If ii, then i; that is:

> If the average of a and b, $\dfrac{(a+b)}{2}$, is larger than a,
> then b is larger then a.

By hypothesis:

$$a < \frac{a+b}{2}.$$

Then,

$$2a < a + b$$

and

$$a < b.$$

So, the statement "If ii, then i" is true.

Part 3. If i, then iii; that is:

> If the number b is larger than the number a,
> then their average is smaller than b.

By hypothesis:

$$a < b.$$

Because we want to obtain $a + b$, we can add either a or b to both sides of the inequality. Because the conclusion we want to reach deals with b, we could try adding b. Thus, we obtain:

$$a + b < 2b.$$

Dividing by 2 yields

$$\frac{a+b}{2} < b.$$

This proves that the conclusion holds true. So, the statement "If i, then iii" is true.

Part 4. If iii, then i; that is:

> If the average of a and b, $\dfrac{(a+b)}{2}$, is smaller than b,
> then b is larger than a.

By hypothesis:

$$\frac{a+b}{2} < b.$$

Then,

$$a + b < 2b$$

and

$$a < b.$$

So, the statement "If iii, then i" is true.

Because statements ii and iii are both equivalent to statement i, they are equivalent to each other. Thus, the proof is now complete. ■

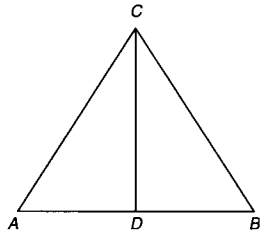
Often we need to prove that two or more definitions of the same object are equivalent. The existence of different definitions is usually generated by different approaches that emphasize a certain property and point of view over another.

Example 4 The following definitions are equivalent:

i. A triangle is an isosceles triangle if it has two equal sides.
ii. A triangle is an isosceles triangle if it has two equal angles.

Proof:

Part 1. If i, then ii.
We have to prove that if a triangle has two equal sides, then it has two equal angles.

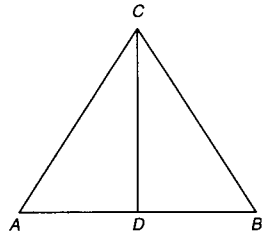


Suppose that the two sides AC and AB are equal. Consider the two triangles ADC and CDB, obtained by constructing the segment CD, perpendicular to the base AB (this is the third side not mentioned in the hypothesis).

The angles $\angle ADC$ and $\angle BDC$ are equal, because they are right angles.

The two triangles have two equal sides: CD, because it is a common side, and AC, which is equal to CB by hypothesis. Thus, AD and DB are equal (we can use the Pythagorean theorem to reach this conclusion). This implies that the triangles ADC and CDB are congruent, and the two angles at the vertices A and B are equal.

Part 2. If ii, then i.
We have to prove that if a triangle has two equal angles, then it has two equal sides.



Suppose that the angles $\angle CAB$ and $\angle CBA$ are equal. Consider the two triangles ADC and CDB, obtained by constructing the segment CD, which starts from the third vertex and is perpendicular to the base, AB.

The angles $\angle ADC$ and $\angle BDC$ are equal, as they are right angles. The two angles at the vertices A and B are equal by hypothesis; therefore, the angles $\angle ACD$ and $\angle DCB$ are equal as well. Thus, the two triangles ADC and CDB are similar.

Moreover they have the side CD in common. Thus, the triangles are congruent. In particular, the sides AC and CB are equal. ■

Example 5. Let f be a positive function defined for all real numbers. Then the following statements are equivalent:

1. f is a decreasing function.
2. The function g, defined as $g(x) = 1/f(x)$, is increasing.
3. The function h, defined as $h(x) = -f(x)$, is increasing.
4. The function k_n, defined as $k_n(x) = nf(x)$, is decreasing for all positive real numbers n.

(See front material in the book for the definitions of increasing and decreasing functions.)

Proof: We will prove that statement 1 implies statement 2, statement 2 implies statement 3, statement 3 implies statement 4, and statement 4 implies statement 1, as shown in the following diagram:

$$\begin{array}{ccc} 1 & \rightarrow & 2 \\ \uparrow & & \downarrow \\ 4 & \leftarrow & 3 \end{array}$$

Part 1. If 1, then 2.

As f is decreasing, given any two real numbers x_1 and x_2 such that $x_1 < x_2$, it follows that $f(x_1) > f(x_2)$. Therefore, $1/f(x_1) < 1/f(x_2)$. This means that $g(x_1) < g(x_2)$, and g is an increasing function.

Part 2. If 2, then 3.

By definition of the functions used, $h(x) = -1/g(x)$. As g is an increasing function, for every two real numbers x_1 and x_2 such that $x_1 < x_2$, it follows that $g(x_1) < g(x_2)$. Thus, $1/g(x_1) > 1/g(x_2)$, and $-1/g(x_1) < -1/g(x_2)$. This implies that $h(x_1) < h(x_2)$, so h is an increasing function.

Part 3. If 3, then 4.

By definition of the functions used, $k_n(x) = n(-h(x)) = -nh(x)$. As h is an increasing function, for every two real numbers x_1 and x_2 such that $x_1 < x_2$, it follows that $h(x_1) < h(x_2)$. Therefore, $-h(x_1) > h(x_2)$, and

$-nh(x_1) > -nh(x_2)$. Thus, $k_n(x_1) > k_n(x_2)$, which proves that k_n is a decreasing function.

Part 4. If 4, then 1.
By definition of the functions used, $f(x) = k_1(x)$. Thus, this implication is trivially true.

The proof is now complete. ■

EXERCISES

Prove the following statements.

1. A function f is nonincreasing if and only if $\frac{f(x)-f(c)}{x-c} \leq 0$ for all c and x in the domain of f with $x \neq c$. (See front material of the book for the definition of nonincreasing function.)
2. The product of two integers is odd if and only if they are both odd.
3. Let n be a positive integer. Then n is divisible by 3 if and only if n^2 is divisible by 3.
4. Let r and s be two counting numbers. The following statements are equivalent:
   i. $r > s$.
   ii. $a^s < a^r$ for all real numbers $a > 1$.
   iii. $a^r < a^s$ for all real positive numbers $a < 1$.
5. Let a and b be two distinct real numbers. The following statements are equivalent:
   i. The number b is larger than the number a.
   ii. Their average, $(a + b)/2$, is larger than a.
   iii. Their average, $(a + b)/2$, is smaller than b.

   Prove this statement by proving "If i, then ii," "If ii, then iii," and "If iii, then i."
6. Let x and y be two distinct negative real numbers. The following statements are equivalent:
   i. $x < y$.
   ii. $|x| > |y|$.
   iii. $x^2 > y^2$.
7. Consider the two systems of linear equations:

$$S_1 \quad \begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

and

$$S_2 \quad \begin{cases} a_1x + b_1y = c_1 \\ (a_1 + ba_2)x + (b_1 + bb_2)y = c_1 + bc_2 \end{cases}$$

where a_1, a_2, b_1, b_2, c_1, and c_2 are real numbers, and $b \neq 0$.
The pair of values (x_0, y_0) is a solution of S_1 if and only if it is a solution of S_2.

USE OF COUNTEREXAMPLES

An example can be very useful when trying to make a point or explain the result obtained in a proof, but it cannot be used as a proof of the fact that a statement is true.

Let us see what might happen if we used and accepted examples as proofs. We could make the claim that if a and b are any two real numbers, then

$$(a + b)^2 = a^2 + b^2.$$

When asked to support our claim, we can produce a multitude of pairs of numbers that satisfy this equality. For example, consider $a = 0$ and $b = 1$:

$$(a + b)^2 = (0 + 1)^2 = 1$$
$$a^2 + b^2 = 0^2 + 1^2 = 1.$$

Thus, it is true that $(a + b)^2 = a^2 + b^2$.

Consider some more examples, such as: $a = 0$ and $b = -1$; $a = -4$ and $b = 0$; $a = \pi$ and $b = 0$; and so on. For all these pairs, the equality $(a + b)^2 = a^2 + b^2$ holds true. But, it is possible to notice that in all the pairs listed above at least one number is equal to zero. The claim states that the equality holds true for *any* two real numbers, not just for some special pairs. What happens if we consider $a = 1$ and $b = 2$?

$$(a + b)^2 = 3^2 = 9$$
$$a^2 + b^2 = 1^2 + 2^2 = 5.$$

Therefore, the equality is false. In spite of all the examples that seem to support it, we have found an example that contradicts it, a *counterexample.*

A *counterexample* is an acceptable proof of the fact that the statement "If **A**, then **B**" is false because it shows that **B** can be and is false while **A**

is true (remember that this situation is the only one for which the statement "If **A**, then **B**" is false). Indeed, to prove that a statement is false, it is enough to prove that it is false in just one instance.

Examples cannot replace the proof that a statement is true *in general*, because examples deal with *special cases*. A counterexample can prove that a statement is false in general, because it exhibits one case in which the statement is false.

Consider the statement "Every real number has a reciprocal." We can think of millions of numbers that do have a reciprocal. But the existence of *one* number with no reciprocal (the number zero does not have a reciprocal) makes the statement "*Every* real number has a reciprocal" false. What is true is the statement "Every *nonzero* real number has a reciprocal."

Sometimes the existence of a counterexample can help us understand why a statement is not true and whether a restriction of the hypothesis (or the conclusion) can change it into a true statement.

The statement "The equality

$$(a+b)^2 = a^2 + b^2$$

holds for all pairs of real numbers a and b in which at least one of the two numbers is zero" is a true statement. (Prove it.)

The discovery of a counterexample can save the time and effort spent trying to construct a proof, but sometimes even counterexamples are not easy to find. Moreover, there is no sure way of knowing when to look for a counterexample. If the best attempts at constructing a proof have failed, then it might make sense to look for a counterexample. This search might be difficult, but if it's successful then it proves that the statement is false. If it is unsuccessful, it will provide examples that support the statement and might give an insight into why the statement is true. And this might give new ideas for the construction of the proof.

Example 1. For all real numbers $x > 0$, $x^3 > x^2$.

Discussion: It might be a good idea to graph the functions x^2 and x^3 to compare them. Let:

A. The number x is a positive real number. (We can use all the properties and operations of real numbers.)

B. $x^3 > x^2$

Proof: Let us look for a counterexample. If $x = 0.5$, then $x^3 = 0.125$ and $x^2 = 0.25$. Therefore, in this case, $x^3 < x^2$. So the statement is false. ■

Note that the statement "For all real numbers $x > 1$, $x^3 > x^2$" is true.

Example 2. If a positive integer number is divisible by a prime number, then it is not prime.

Proof: The statement is false. Consider the prime number 7. It is a positive integer number and it is divisible by the prime number 7 (indeed $7/7 = 1$). So, it satisfies the hypothesis, but 7 is a prime number. Thus, the conclusion is false. ■

The statement "If a positive integer number is divisible by a prime number and the quotient of the division is not 1, then it is not prime" is true.

Example 3. If an integer is a multiple of 10 and 15, then it is a multiple of 150.

Proof: The statement is false. Just consider the least common multiple of 10 and 15, namely 30. This number is a multiple of 10 and 15, but it is not a multiple of 150. ■

The statement "If an integer is a multiple of 10 and 15, then it is a multiple of 30" is true.

EXERCISES

Use counterexamples to prove that the following statements are false.

1. Let f be an increasing function and g be a decreasing function. Then the function $f+g$ is constant. (See front material of the book for the definitions of nonincreasing function and $f+g$.)
2. If t is an angle in the first quadrant, then $2\sin t = \sin 2t$.
3. Consider the polynomial $P(x) = -x^2 + 2x - 3/4$. If $y = P(x)$, then y is always negative.
4. The reciprocal of a real number $x \geq 1$ is a number y such that $0 < y < 1$.
5. The number $2^n + 1$ is prime for all counting numbers n.
6. Let f, g, and h be three functions defined for all real numbers. If $f \circ g = f \circ h$, then $g = h$.

Discuss the truth of the following statements; that is, prove those that are true and provide counterexamples for those that are false.

7. The sum of any five consecutive integers is divisible by 5.
8. If $f(x) = x^2$ and $g(x) = x^4$, then $f(x) \leq g(x)$ for all real numbers $x \geq 0$.
9. The sum of four consecutive counting numbers is divisible by 4 (see exercise 7).

10. Let f and g be two odd functions defined for all real numbers. Their sum, $f+g$, is an even function defined for all real numbers. (See front material of the book for the definitions of even and odd functions and $f+g$.)
11. Let f and g be two odd functions defined for all real numbers. Then their quotient function f/g is an even function defined for all real numbers. (See front material of the book for the definitions of even and odd functions.)
12. A six-digit palindrome number is divisible by 11.
13. The sum of two numbers is a rational number if and only if both numbers are rational.
14. Let f be an odd function defined for all real numbers. The function $g(x) = (f(x))^2$ is even. (See front material of the book for the definitions of even and odd functions.)
15. Let f be a positive function defined for all real numbers. The function $g(x) = (f(x))^3$ is always increasing. (See front material of the book for the definitions of increasing function.)

MATHEMATICAL INDUCTION

In general, we use this kind of proof when we need to show that a certain statement is true for an infinite collection of natural numbers, and direct verification is impossible. We cannot simply check that the statement is true for some of the numbers in the collection and then generalize the result to the whole collection. Indeed, if we did this, we would just provide examples, and, as already mentioned several times, examples are not proofs.

Consider the following claim: The inequality

$$n^2 \leq 5n!$$

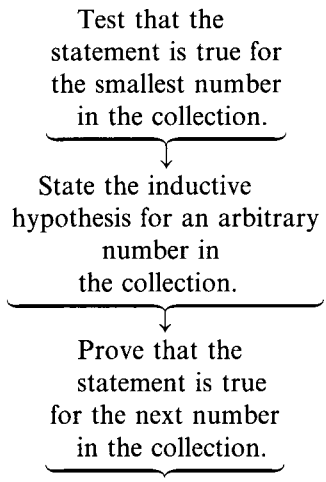
is true for all counting numbers $n \geq 3$.

How many numbers should we check? Is the claim true because the inequality holds true for $n = 3, 4, 5, 6, \ldots, 30$? We cannot check directly all counting numbers $n \geq 3$. Therefore, we must look for another way to prove this kind of statement.

The technique of proving a statement by using *mathematical induction* (*complete induction*) consists of the following three steps:

1. Prove that the statement is true for the smallest number included in the statement to be proved (base case).
2. Assume that the statement is true for an arbitrary number in the collection (inductive hypothesis).

3. Using the inductive hypothesis, prove that the statement is true for the next number in the collection (deductive step).


Test that the statement is true for the smallest number in the collection.
State the inductive hypothesis for an arbitrary number in the collection.
Prove that the statement is true for the next number in the collection.

At first, the construction of the proof by induction might seem quite peculiar. We start by checking that the given statement is true in a special case. We know that we cannot stop here because examples are not proofs. Then we seem to "trust" the statement to be true temporarily, and then we check its strength by using deductive reasoning to see if the truth of the result can be extended one step further. If the statement passes this last test, then the proof is complete. This construction works like a row of dominoes; when the first one is knocked down, it will knock down the one after it, and so on until the entire row is down.

The three steps show that the statement is true for the first number and that whenever the statement is true for a number it will be true for the next. The fact that this extension process can be extended indefinitely from the smallest number requires an in-depth explanation that is beyond the purpose of this book. Indeed, the technique of mathematical induction is founded on a very important theoretical result—namely, the *principle of mathematical induction*, usually stated as follows:

Let $P(n)$ represent a statement relative to a positive integer n. If:

1. $P(t)$ is true, where t is the smallest integer for which the statement can be made,
2. whenever $P(n)$ is true, it follows that $P(n+1)$ is true as well,

then $P(n)$ is true for *all* $n \geq t$.

Example 1. Prove by induction that the sum of the first k natural numbers is equal to $k(k+1)/2$.

Proof: We want to prove that the equality:

$$\underbrace{1+2+3+\cdots+(k-1)+k}_{\text{k numbers}} = \frac{k(k+1)}{2}$$

holds true for all $k \geq 1$ natural numbers.

1. *Base case.* Does the equality hold true for $k=1$, the smallest number that can be used?

$$1 = \frac{1(1+1)}{2}.$$

Thus, by using the given formula we obtain a true statement. This means that the formula works for $k=1$.

2. *Inductive hypothesis.* Assume the formula works when we add the first n numbers ($k=n$). Thus,

$$\underbrace{1+2+3+\cdots+(n-1)+n}_{\text{n numbers}} = \frac{n(n+1)}{2}.$$

3. *Deductive proof.* We want to prove that the formula holds true for the next number, $n+1$. Thus, we have to prove that:

$$\underbrace{1+2+3+\cdots+n+(n+1)}_{(n+1)\text{ numbers}} = \frac{(n+1)[(n+1)+1]}{2}$$

or, equivalently,

$$\underbrace{1+2+3+\cdots+n+(n+1)}_{(n+1)\text{ numbers}} = \frac{(n+1)(n+2)}{2}.$$

To reach this goal, we will need to use the equality stated in the inductive hypothesis:

$$1+2+3+\cdots+n+(n+1)$$

associative property of addition of numbers:

$$= [1+2+3+\cdots+n]+(n+1)$$

use the equality stated in the inductive hypothesis:

$$= \frac{n(n+1)}{2} + (n+1)$$

perform algebraic steps:

$$= \frac{n(n+1) + 2(n+1)}{2}$$
$$= \frac{(n+1)(n+2)}{2}.$$

Thus,

$$\underbrace{1+2+3+\cdots+n+(n+1)}_{(n+1) \text{ numbers}} = \frac{(n+1)(n+2)}{2}$$

Therefore, the formula given in the statement holds true for all natural numbers $k \geq 1$, by the principle of mathematical induction. ■

A different proof of the result stated in Example 1 can be found in the previous chapter (see Example 3). This is one of the nice situations in which several proofs of the same statement can be constructed using different mathematical tools.

Example 2. The sum of the first k odd numbers is equal to k^2; that is:

$$1+3+5+\cdots+(2k-1) = k^2$$

Proof:

1. *Base case.* Does the equality hold true for $k=1$, the smallest number that can be used? In this case, we are considering only one odd number. Therefore, we have:

   $$1 = 1^2$$

   Thus, the equality is true for $k=1$.

2. *Inductive hypothesis.* Assume the equality holds true for an arbitrary collection of n odd numbers. Thus,

   $$1+3+5+\cdots+(2n-1) = n^2.$$

3. *Deductive proof.* We want to prove that the equality is true for $n+1$ odd numbers. Therefore, we need to check the equality:

   $$1+3+5+\cdots+(2n-1)+[2(n+1)-1] = (n+1)^2.$$

Using the associative property of addition, we can write:

$$\begin{aligned} &1 + 3 + 5 + \cdots + (2n - 1) + [2(n + 1) - 1] \\ &= \{1 + 3 + 5 + \cdots + (2n - 1)\} + (2n + 1). \end{aligned}$$

Then we can use the inductive hypothesis to obtain:

$$\begin{aligned} &1 + 3 + 5 + \cdots + (2n - 1) + [2(n + 1) - 1] \\ &= n^2 + (2n + 1) \\ &= (n + 1)^2. \end{aligned}$$

Therefore, the formula given in the statement holds true for the sum of an arbitrary number of odd natural numbers, by the principle of mathematical induction. ■

The principle of mathematical induction stated before Example 1 is also known as the *weak (or first) principle of mathematical induction*, in contrast to the *strong (or second) principle of mathematical induction*, usually stated as follows:

Let $P(n)$ represent a statement relative to a positive integer n. If:

1. $P(t)$ is true, where t is the smallest integer for which the statement can be made,
2. whenever $P(k)$ is true for all numbers k with $k = t, t + 1, \ldots, n$, it follows that $P(n + 1)$ is true as well,

then $P(n)$ is true for *all* $n \geq t$.

Let us see how to use this principle in a proof.

Example 3. If $n > 1$ is a counting number, then either n is a prime number or it is a product of primes.

Proof:

1. *Base case.* The statement is true for the smallest number we can consider, which is 2. This number is indeed prime.
2. *Inductive hypothesis.* Assume the statement is true for all the numbers between 2 and an arbitrary number k, including k.
3. *Deductive proof.* Is the statement true for $k + 1$?
   If $k + 1$ is a prime number, then the statement is trivially true.
   If $k + 1$ is not a prime number, then by definition it has a positive divisor d such that $d \neq 1$ and $d > k + 1$.
   Thus, $k + 1 = dm$ with $m \neq 1$, $m > k + 1$, and $m \geq 2$.

As both d and m are positive numbers larger than or equal to 2 and smaller than or equal to k, by inductive hypothesis they are either primes or product of prime numbers.
If they are both prime numbers, the statement is proved.
If at least one of them is not prime, we can replace it with its prime factors. So $k+1$ will be a product of prime factors in any case.

Therefore, by the second principle of mathematical induction, the statement is true for all natural numbers $n > 1$. ■

The adjectives "strong" and "weak" attached to the two statements of the principle of mathematical induction are not always used in the same way by different authors and they are misleading. When reading the two statements, it is easy to see that the *strong principle of mathematical induction* implies the *principle of mathematical induction.* Indeed, the inductive hypothesis of the *principle of mathematical induction* assumes that the given statement is true just for an arbitrary number n. The inductive hypothesis of the *strong principle of mathematical induction* assumes that the given statement is true for all the numbers between the one covered by the base case and an arbitrary number n.

In reality, *these two principles are equivalent.* The proof of this claim is not easy; it consists of proving that both principles are equivalent to a third principle, the well-ordering principle (see the material in the front of the book). So, in turn, the *strong principle of mathematical induction* and the *principle of mathematical induction* are equivalent to each other.

Then which one of the two principles should be used in a proof by induction? The first answer is that really it does not make any difference, as one could always use the *strong principle of mathematical induction.* in general, when proving equalities, the *principle of mathematical induction* is sufficient. For some mathematicians, it is a matter of elegance and beauty to use as little machinery as possible when constructing a proof; therefore, they prefer to use the *principle of mathematical induction* whenever possible.

We will consider another example that requires the use of the strong principle of mathematical induction.

Example 4. A polynomial of degree $n \geq 1$ with real coefficients has at most n real zeroes, not all necessarily distinct.

Proof:

1. *Base case.* The statement is true for the smallest number we can consider, which is 1. Indeed a polynomial of degree 1 is of the form $P(x) = a_1 x + a_0$ with $a_1 \neq 0$. Its zero is the number $x = -a_0/a_1$. So, a polynomial of degree 1 with real coefficients has one real zero.

2. *Inductive hypothesis.* Assume the statement is true for all polynomials with real coefficients whose degree is between 2 and an arbitrary number k, including k.
3. *Deductive proof.* Is the statement true for polynomials with real coefficients of degree $k+1$? Let $Q(x) = a_{k+1}x^{k+1} + a_k x^k + \cdots + a_1 x + a_0$ be one of such polynomials.

   If $Q(x)$ has no real zeroes, then the statement is true.
   Assume that $Q(x)$ has at least one real zero, c, which could be a repeated zero.
   Then using the rules of algebra we can write:

$$Q(x) = (x - c)^t T(x)$$

   where $t \geq 1$ and $T(x)$ is a polynomial of degree $(k+1) - t$, with real coefficients.
   If $t = k+1$, then $T(x)$ is a constant, and the statement is true because $Q(x)$ has exactly $k+1$ coincident zeroes.
   If $t < k+1$, then $T(x)$ is a polynomial of degree between 1 and k, with real coefficients. So, by the inductive hypothesis and the base case, $T(x)$ has at most $(k+1) - t$ real zeroes.
   As every zero of $T(x)$ is a zero of $Q(x)$, as well, all the zeroes of $T(x)$ must be counted as zeroes of $Q(x)$.
   Thus, the real zeroes of $Q(x)$ are all the real zeroes of $T(x)$ and c, which might be counted as a zero exactly t times. So, $Q(x)$ has at most $\{[(k+1) - t] + t\} = k+1$ real zeroes.

Therefore, by the strong principle of mathematical induction, the statement is true for all $n \geq 1$. ■

One important comment regarding this kind of proof (by mathematical induction) is that the inductive hypothesis *must* be used in the construction of the proof of the last step. If this does not happen, then we are either using the wrong technique or making a mistake in the construction of the deductive step. An illustration of this is given in the next example.

Example 5. The difference of powers $7^k - 4^k$ is divisible by 3 for all $k \geq 1$.

Incorrect Use of Proof by Induction

1. *Base case.* The statement is true for the smallest number we can consider, which is 1, because $7^1 - 4^1 = 3$.
2. *Inductive hypothesis.* Assume the statement is true for an arbitrary number n; that is, $7^n - 4^n = 3t$ for an integer number t.
3. *Deductive proof.* Is the statement true for $n+1$? Is $7^{n+1} - 4^{n+1} = 3s$ for an integer number s?

Using the factorization technique for differences of two powers (this is possible because $n+1 \geq 2$), we can write:

$$\begin{aligned} 7^{n+1} - 4^{n+1} &= (7-4)(7^n + 7^{n-1} \times 4 + \cdots + 7 \times 4^{n-1} + 4^n) \\ &= 3(7^n + 7^{n-1} \times 4 + \cdots + 7 \times 4^{n-1} + 4^n). \end{aligned}$$

As the number in parentheses is an integer (it is a combination of integers), we proved that $7^{n+1} - 4^{n+1} = 3s$ for an integer number s.

Clearly, in the proof of the deductive step we have not used the inductive hypothesis to reach the conclusion. This might imply either that the proof of the original statement can be constructed without using the principle of mathematical induction (in which case we need to redo the entire proof to make sure that the logic of it is correct) or that we made a mistake in the third step (in which case we can work on it and still use the previous two steps).

Correct Proof by Induction

1. *Base case.* The statement is true for the smallest number we can consider, which is 1, because $7^1 - 4^1 = 3$.
2. *Inductive hypothesis.* Assume the statement is true for an arbitrary number n; that is $7^n - 4^n = 3t$ for an integer number t.
3. *Deductive proof.* Is the statement true for $n+1$? Is $7^{n+1} - 4^{n+1} = 3s$ for an integer number s?

   We will use properties of exponents and other rules of algebra to obtain the expression $7^n - 4^n$ so we can use the inductive hypothesis. Thus, we have:

$$\begin{aligned} 7^{n+1} - 4^{n+1} &= 7 \times 7^n - 4 \times 4^n \\ &= (3+4) \times 7^n - 4 \times 4^n = 3 \times 7^n + 4 \times 7^n - 4 \times 4^n \\ &= 3 \times 7^n + 4 \times (7^n - 4^n). \end{aligned}$$

   At this point we can use the inductive hypothesis to write:

$$\begin{aligned} 7^{n+1} - 4^{n+1} &= 3 \times 7^n + 4 \times (7^n - 4^n) \\ &= 3 \times 7^n + 4 \times 3t = 3(7^n + 4t). \end{aligned}$$

   Because the number $7^n + 4t$ is an integer number, we proved that $7^{n+1} - 4^{n+1} = 3s$ for an integer number s.

Therefore, by the principle of mathematical induction the original statement is true. ■

Another Correct Proof

1. *Case 1.* The statement is true for $k = 1$, because $7^1 - 4^1 = 3$.
2. *Case 2.* Let $k > 1$. Then it is possible to write $k = n + 1$, with $n \geq 1$. Using the factorization technique for differences of two powers, we can use the following equality:

$$\begin{aligned} 7^{n+1} - 4^{n+1} &= (7-4)(7^n + 7^{n-1} \times 4 + \cdots + 7 \times 4^{n-1} + 4^n) \\ &= 3(7^n + 7^{n-1} \times 4 + \cdots + 7 \times 4^{n-1} + 4^n). \end{aligned}$$

Because the number in parentheses is an integer, we proved that $7^k - 4^k = 3s$ for an integer number s. Thus, the number $7^k - 4^k$ is divisible by 3. ■

From a technical point of view, one could argue that the factorization formula used to factor $7^n - 4^n$ must be proved using mathematical induction, so we have a completely self-contained proof that does not invoke factorization techniques established in some other context. We trust the reader to be familiar with them.

Statements involving inequalities are, in general, more difficult to prove than those involving equalities. Indeed, when working with inequalities, there is a certain degree of freedom because the relation between the mathematical expressions involved is not quite as unique as the one determined by equality. A mathematical expression can only be equal to some "variation" of itself, so all we can do is try to rewrite it in seemingly different ways using the rules of algebra. But, a mathematical expression can be larger (or smaller) than several other expressions, and we need to choose the one that is useful for completing our task.

Example 6. For all integer numbers $a \geq 4$:

$$a^3 \geq 3a^2 + 3a + 1$$

Proof: We will use the principle of mathematical induction.

1. *Base case.* The statement is true for the smallest number we can consider, namely 4, because $4^3 \geq 3 \times 4^2 + 3 \times 4 + 1$, since $64 \geq 61$.
2. *Inductive hypothesis.* Assume the statement is true for an arbitrary number n; that is,

$$n^3 \geq 3n^2 + 3n + 1.$$

3. *Deductive proof.* Is the statement true for $n + 1$? Is

$$(n+1)^3 \geq 3(n+1)^2 + 3(n+1) + 1?$$

If we simplify the right-hand side of the inequality, we can rewrite it as

$$(n+1)^3 \geq 3n^2 + 9n + 7.$$

By finding the third power of the binomial $n+1$ and using the inductive hypothesis to replace n^3 we obtain

$$\begin{aligned}(n+1)^3 &= n^3 + 3n^2 + 3n + 1 \\ &\geq (3n^2 + 3n + 1) + 3n^2 + 3n + 1.\end{aligned}$$

Combining some of the similar terms yields

$$(n+1)^3 \geq 3n^2 + 3n^2 + 6n + 2.$$

At this point, there are several ways to proceed. Thus, it is very important to keep in mind the conclusion we want to reach. Because $n \geq 4$, $3n^2 = 3n \times n \geq 12n$. Therefore,

$$\begin{aligned}(n+1)^3 &\geq 3n^2 + 3n^2 + 6n + 2 \\ &\geq 3n^2 + 12n + 6n + 2 \\ &= 3n^2 + 9n + (9n + 2).\end{aligned}$$

Again, $n \geq 4$ implies $9n + 2 \geq 7$. So,

$$(n+1)^3 \geq 3n^2 + 9n + 7.$$

This is exactly the conclusion we wanted to reach. Therefore, by the principle of mathematical induction the inequality holds true for all integers greater than or equal to 4. ■

EXERCISES

Prove the following statements:

1. For all positive integers k,

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1.$$

2. The number $9^k - 1$ is divisible for all $k \geq 1$.
3. For all integer numbers $k \geq 1$

$$2 + 4 + 6 + \cdots + 2k = k^2 + k.$$

4. Let $a > 1$ be a fixed number. Then for all integer numbers $k \geq 3$,

$$(1 + a)^k > 1 + ka^2.$$

5. Check that for all counting numbers $k \geq 1$,

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^k = \frac{1 - \left(\frac{1}{2}\right)^{k+1}}{1 - \frac{1}{2}} - 1.$$

6. The inequality $k^2 \leq 5k!$ is true for all counting numbers $k \geq 3$.
7. Let n be an odd counting number. Then $n^2 - 1$ is divisible by 4.
8. The number $10^k - 1$ is divisible by 9 for all $k \geq 1$.
9. Let n be an integer larger than 4. The next to the last digit from the right of 3^n is even.
10. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $A^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for all $n \geq 2$. (*)

EXISTENCE THEOREMS

Existence theorems are easy to recognize because they claim that at least one object having certain properties exists. Usually this kind of theorem is proved in one of two ways.

1. If it is possible, use an algorithm (a procedure) to construct explicitly at least one object with the required properties.
2. Sometimes, especially in more advanced mathematical theories, the explicit construction is not possible; therefore, we must be able to find a general argument that guarantees the existence of the object under consideration, without being able to provide an actual example of it.

EXAMPLE 1. Given any two distinct rational numbers, there exists a third distinct rational number between them.

Discussion: We will reword the statement above:

A. Consider two distinct rational numbers, a and b, with $a > b$.

Implicit hypothesis: Because the two numbers are not equal, we can assume without loss of generality that one is smaller than the other. We can use all the properties and operation of rational numbers.

B. There exists a rational number c such that $a < c < b$.

Proof: If a and b are two rational numbers, we can write $a = \frac{m}{n}$ and $b = \frac{p}{q}$, where m, n, p, and q are integer numbers; $n \neq 0$ and $q \neq 0$. As the average of two numbers is always between the two of them (see Example 3 in the section on Equivalence Theorems), we can construct the average of a and b and check whether it is a rational number. Thus, we have:

$$c = \frac{a+b}{2} = \frac{1}{2}\left(\frac{m}{n} + \frac{p}{q}\right) = \frac{mq + np}{2nq}.$$

The number $mq + np$ is an integer because m, n, p, and q are integer numbers; the number $2nq$ is a nonzero integer because n and q are integer numbers; $n \neq 0$; and $q \neq 0$. Therefore, the number c is a rational number.

For the proof of the fact that $a < c < b$, see the section on Equivalence Theorems (Example 3). ■

Example 2. Let x be an irrational number. Then there is at least one digit that appears infinitely many times in the decimal expansion of x.

Discussion: Because we do not know what the number x is we cannot even hope to find explicitly which one of the possible ten digits is repeated infinitely many times. Thus, the proof will not be a constructive one.

Proof: Using the contrapositive of the original statement, let us assume that none of the decimal digits repeats infinitely many times. So, we can assume that each digit k, with $0 \leq k \leq 9$, repeats n_k times. Then the decimal expansion of x has $N = n_0 + n_1 + \cdots + n_8 + n_9$ digits. Therefore, x is a rational number. Because the contrapositive of the original statement is true, the statement itself is true. ■

In Example 1, we were able to construct explicitly the mathematical object that satisfies the given requirements. In Example 2, we can only establish that an object exists, but we cannot explicitly give the procedure for finding it.

Example 3. A line passes through the points with coordinates $(0, 2)$ and $(2, 6)$.

Discussion: We will reformulate the statement as: If $(0, 2)$ and $(2, 6)$ are two points in the plane, then there is a line passing through them. We will prove this statement in two ways: by finding the equation of the line explicitly, and by using a theoretical argument.

Proof:

1. The point–slope equation of a line is:

$$y - y_0 = m(x - x_0)$$

where m is the slope and (x_0, y_0) are the coordinates of any point on the line.

If (x_1, y_1) and (x_0, y_0) are the coordinates of any two points on the line, and $x_1 \neq x_0$, then:

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

Using the points with coordinates $(0, 2)$ and $(2, 6)$, we obtain $m = 4/2 = 2$.

Therefore, the line that passes through the given points has the equation:

$$y - 2 = 2(x - 0)$$

or

$$y = 2x + 2. \quad \blacksquare$$

2. There is a postulate from geometry that states that given any two distinct points in the plane there is a unique straight line joining them. Therefore, there is a line joining the points with coordinates $(0, 2)$ and $(2, 6)$. $\blacksquare$

Example 4. The polynomial $P(x) = x^4 + x^3 + x^2 + x - 1$ has a real zero in the interval $[0, 1]$.

Discussion: If we want to find an explicit value of x such that $P(x) = 0$, we need to solve a fourth-degree equation. This can be done, but the formulas used to solve a fourth-degree equation are quite cumbersome, even if they are not difficult. It is possible to use a calculator, or any numerical method (such as Newton's method) as well, but the statement does not ask us *to find* the real zeroes of the polynomial $P(x)$. We are only asked to prove that one of the zeroes is in the interval $[0, 1]$. The proof that follows requires some knowledge of calculus.

Proof: Polynomials are continuous functions. Because $P(0) = -1$ is a negative number and $P(1) = 3$ is a positive number, by the intermediate value theorem there will be at least one value of x in the interval $[0, 1]$ for which $P(x) = 0$. Thus, the given statement is true. $\blacksquare$

EXERCISES

Prove the following statements.

1. There exists a function whose domain consists of all the real numbers and whose range is in the interval $[0, 1]$.
2. There is a counting number n such that $2^n + 7^n$ is a prime number.
3. Let a be an irrational number. Then there exists an irrational number b such that ab is an integer.
4. There is a second degree polynomial P such that $P(0) = -1$ and $P(-1) = 2$.
5. There exist two rational numbers a and b such that a^b is a positive integer and b^a is a negative integer.
6. If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, with n odd, then the equation $P(x) = 0$ has at least one real solution. $(*)$
7. Let a and b be two rational numbers, with $a < b$. Then there exist at least three rational numbers between a and b.
8. There exists an integer number k such that $2^k > 4^k$.

UNIQUENESS THEOREMS

This kind of theorem states that an object having some required properties, and whose existence has already been established, is unique. In order to prove that this is true, we have to prove that no other object satisfies the properties listed. Direct and explicit checking is usually impossible, because we might be dealing with infinite collections of objects. Therefore, we need to use a different approach. Usually this kind of theorem is proved in one of the three following ways:

1. What would happen if the object with the required properties is not unique? To deny that something is unique means to assume that there is at least one more object with the same properties. So, assume that there are two objects satisfying the given properties, and then prove that they coincide.
2. If the object has been explicitly constructed using an algorithm (a procedure), we might be able to use the fact that every step of the algorithm could only be performed in a unique way.
3. Especially when the explicit construction of the object is not possible, we might be able to find a general argument that guarantees the uniqueness of the object with the required properties.

Example 1. The number, usually indicated by 1, such that:

$$a \times 1 = 1 \times a = a$$

for all real numbers a is unique. (The number 1 is called the *identity* for multiplication of real numbers.)

Proof: Let t be a number with the property that:

$$a \times t = t \times a = a$$

for all real numbers a (even for $a = 1$ and for $a = t$). (We cannot use the symbol 1 for this number, because as far as we know t could be different from 1.)

Because 1 leaves all other numbers unchanged when multiplied by them, we have

$$1 \times t = t.$$

Because t leaves all other numbers unchanged when multiplied by them, we have:

$$1 \times t = 1.$$

Therefore,

$$t = 1 \times t = 1.$$

This proves that $t = 1$. Thus, it is true that only the number 1 has the required properties *(i.e.,* the identity element for multiplication is unique). ■

Very often existence and uniqueness theorems are combined in statements of the form: "There exists a unique ..." The proof of this kind of statements has two parts:

1. Prove the existence of the object described in the statement.
2. Prove the uniqueness of the object described in the statement.

While these two steps can be performed in any order, it seems to make sense to prove the existence of an object before proving its uniqueness. After all, if the object does not exist, its uniqueness becomes irrelevant. If the object can be constructed explicitly (to prove its existence), the steps used in the construction might provide a proof of its uniqueness.

Example 2. The function $f(x) = x^3$ has a unique inverse function.

Proof: We start by recalling that two functions, f and g, are inverse of each other if

$$f \circ g(x) = f(g(x)) = x$$
$$g \circ f(x) = g(f(x)) = x$$

for all real numbers x (because f in this case is defined for all real numbers and its range is the collection of all real numbers).

Part 1. The inverse function of f exists.
Because the function f is described by an algebraic expression, we will look for an algebraic expression for its inverse, g.

The function g has to be such that:

$$f \circ g(x) = f(g(x)) = x.$$

Therefore, using the definition of f we obtain

$$(g(x))^3 = x.$$

This implies that

$$g(x) = x^{1/3} = \sqrt[3]{x}.$$

We need to check that the function obtained in this way is really the inverse function of f. Because

$$f \circ g(x) = f(\sqrt[3]{x}) = \left(\sqrt[3]{x}\right)^3 = x$$
$$g \circ f(x) = g(x^3) = \sqrt[3]{x^3} = x$$

we can indeed conclude that g is the inverse function of f.

Part 2. The inverse function of f is unique.
In this case, we can establish the uniqueness of g in two ways:

a. The function g is unique because of the way it has been found and defined.
b. Let us assume that there exists another function, h, that is the inverse of f. Then, by definition of inverse,

$$h \circ f(x) = x$$
$$f \circ h(x) = x$$

for all real numbers x.

We want to compare the two functions g and h. They are both defined for all real numbers as they are inverses of f. To compare them, we have to compare their outputs for the same value of the variable. While we have a formula for g, we do not have a formula for h. So we need to use the *properties* of h and g:

$$\begin{aligned} g(x) &= g(f \circ h(x)) = g(f(h(x))) \\ &= (g \circ f)(h(x)) = h(x). \end{aligned}$$

Therefore, $g = h$. So, the inverse of f is unique. ■

A shorter and less explicit proof of the existence part of the statement in Example 2 relies on a broader knowledge of functions and inverse function. We will mention it for sake of completeness. The function f is one-to-one and onto; therefore, it will have an inverse function. See proof 1 in the Exercises for this section.

Example 3. We proved that if n is an integer number larger than 1, then n is either prime or a product of prime numbers. Thus, we can write:

$$n = p_1 \times p_2 \times \cdots \times p_k$$

where the p_j are prime numbers, and $p_1 \leq p_2 \leq \cdots \leq p_k$. This factorization of n is unique.

Proof: Let us assume that there are at least two ways of writing n as the product of prime factors listed in nondecreasing order. Therefore,

$$p_1 \times p_2 \times \cdots \times p_k = n = q_1 \times q_2 \times \cdots \times q_s.$$

Thus, the prime factor p_1 divides the product $q_1 \times q_2 \times \cdots \times q_s$ (indeed $q_1 \times q_2 \times \cdots \times q_s / p_1 = p_2 \times p_3 \times \cdots \times p_k$). This implies that p_1 divides at least one of the q_j. Let us assume that p_1 divides q_1 (we can reorder the q_j).

As q_1 is prime, this implies that $p_1 = q_1$. Therefore, after simplifying p_1 and q_1, we have:

$$p_2 \times \cdots \times p_k = q_2 \times \cdots \times q_s.$$

Similarly, p_2 divides $q_2 \times \cdots \times q_s$. So, we can assume that p_2 divides q_2. This again implies that $p_2 = q_2$. Thus,

$$p_3 \times \cdots \times p_k = q_3 \times \cdots \times q_s.$$

So, if $k < s$, we obtain $1 = q_{k+1} \times \cdots \times q_s$.

This equality is impossible because all the q_j are larger than 1 (they are prime numbers).

If $k > s$, we obtain $1 = p_{s+1} \times \cdots \times p_k$. Again, this is impossible. Therefore, $k = s$ and $p_1 = q_1, p_2 = q_2, \ldots, p_k = q_k$. Therefore, the factorization of n is unique for the prime numbers used. The order in which these factors are arranged is unique, as it is fixed. Therefore, the factorization of n as described is unique. ■

EXAMPLE 4. There exists a unique line passing through the points with coordinates (0, 2) and (2, 6).

Proof:

Part 1. There exists a line passing through the points with coordinates (0, 2) and (2, 6).

See Example 3 in the section on Existence Theorems.

Part 2. We will prove the uniqueness of the line using all three procedures described at the beginning of the section.

First procedure. Assume that there are two lines passing through the points with coordinates (0, 2) and (2,6). Let their equations be $y = ax + b$ and $y = cx + d$.

As both lines pass through the point (0,2), we have $0a + b = 2$, and $0c + d = 2$. Thus, $b = d = 2$.

As both lines pass through the point (2,6), we have $2a + b = 2c + d$. Because $b = d$, this implies that $a = c$. Thus, the two lines coincide.

Second procedure. If we look at the steps used to find the equation of the line (refer to Example 3 in the section on Existence Theorems) as $y = 2x + 2$, we can state that:

1. The slope is uniquely determined by the coordinates of the points; and
2. Given the uniqueness of the slope, the other constant in the formula is uniquely determined as well.

Third procedure. There is a postulate from geometry that states that given any two distinct points in the plane there is a *unique* straight line joining them. Therefore, there is a unique line joining the points with coordinates (0, 2) and (2, 6). ■

EXERCISES

Prove the following statements.

1. The polynomial $p(x) = x - b$ has a unique solution for all real numbers b.
2. There exists a unique angle θ with $0 \leq \theta \leq \pi$ such that $\cos \theta = \theta$.

3. The equation $x^3 - b = 0$, with b real number, has a unique solution.
4. There exists a unique second degree polynomial P such that $P(0) = -1$, $P(1) = 3$, and $P(-1) = 2$.
5. The graphs of the functions $f(x) = x^3$ and $g(x) = -x^2 - 2x$ have a unique intersection point.

Outline the proofs of the following statements, filling in all details as needed:

1. Let f be a function defined for all real numbers. If f is one-to-one and onto, its inverse exists and it is unique. (See the front material of the book for the definition of one-to-one and onto.)

   Proof: We need to find a function g defined for all real numbers such that:

   $$g \circ f(x) = g(f(x)) = x$$
   $$f \circ g(y) = f(g(y)) = y$$

   for all real numbers x and y, where x indicates a number in the domain of f, and y indicates a number in the range of f.
   The function g is defined as follows. Let y be any real number, then $g(y) = x$, where x is the unique number with the property that $f(x) = y$.
   (How do we know that such a number x exists and it is unique for any value of y?)
   We claim that the function g defined in this way is indeed the inverse of the function f.
   Check: Let x_0 be a real number such that $f(x_0) = y_0$. Thus,

   $$g \circ f(x_0) = g(f(x_0)) = g(y_0) = x_0.$$

   On the other hand, if $g(y_0) = x_0$, it follows that $f(x_0) = y_0$. So,

   $$f \circ g(y_0) = f(g(y_0)) = f(x_0) = y_0.$$

   Assume that there is another function h that is the inverse of f. Therefore,

   $$h \circ f(x) = x$$
   $$f \circ h(y) = y$$

   for all real numbers x and y. Thus,

   $$f \circ h(y) = y = f \circ g(y)$$

for all real numbers y. This implies that:

$$f(h(y)) = f(g(y)).$$

Because f is a one-to-one function, we obtain:

$$h(y) = g(y)$$

for all real numbers y. Therefore, $h = g$. ■

2. *Division theorem.* Let a and b be two integer numbers such that $a \geq 0$ and $b > 0$. Then, there exist two unique integers q and r, where $q \geq 0$ and $0 \leq r < b$ such that $a = bq + r$.
(We have already proved the existence of numbers with the required properties earlier in the book; see Example 5 in the section on Basic Techniques. Here, we want to include a different proof that uses mathematical induction.)

   Proof: Consider different cases and use mathematical induction.

   1. If $b = 1$, consider $q = a$ and $r = 0$.
   2. If $a = 0$, consider $q = 0$ and $r = 0$.
   3. The statement is clearly true for all numbers $a < b$. Consider $q = 0$ and $r = a$.
   4. Assume that $a \geq b$.

   If $a = b$, the statement is trivially true.
   Assume that the statement is true for a generic number $n > b$. Then consider $a = n + 1$.

$$\begin{aligned} a = n + 1 &= (bq_1 + r_1) + 1 \\ &= bq_1 + (r_1 + 1). \end{aligned}$$

   Because $0 \leq r_1 < b$, then $1 \leq r_1 + 1 \leq b$.
   If $r_1 + 1 < b$, then just set $q_1 = q$ and $r_1 + 1 = r$.
   If $r_1 + 1 = b$, we can write

$$a = bq_1 + b = bq + r$$

   where $q = q_1 + 1$ and $r = 0$.
   We are now going to prove the uniqueness part of the statement. Let us assume that there are two pairs of integers, q and r, q' and r' such that

$$a = bq + r = bq' + r'$$

where $q \geq 0$ and $0 \leq r < b$, and $q' \geq 0$ and $0 \leq r' < b$.
Without loss of generality, we can assume that $r' \geq r$.
Thus,

$$b(q - q') = r' - r \geq 0.$$

Because $r' > r' - r$, we have $b > r' - r$.
Therefore,

$$b > b(q - q') \geq 0.$$

Dividing by b we obtain

$$1 > q - q' \geq 0.$$

This implies that $q - q' = 0$, or $q = q'$.
If $q - q' = 0$, then $r' - r = 0$. So, $r' = r$.
The theorem is now completely proved.

EQUALITY OF SETS

A *set* is a well-defined collection of objects. The objects that belong to a set are called the *elements* of the set. If x is an element of a set A, we write $x \in A$.

The *empty set* is a set with no elements, usually represented by either $\emptyset$ or $\{\}$.

Sets can be described in several ways. We can either provide a list of the elements (roster method) or we can list the property (properties) the elements must have in order to belong to the set (constructive method). If we are listing the elements of a set, the order of the listing is irrelevant, and the same element should appear only once (repeated elements do not count as distinct elements).

The roster method is not very practical when the set has a large number of elements, and it is impossible to use when the set has an infinite number of elements. In this latter case, we should list enough elements for a pattern to emerge and then use "..."; for example, it would be a bad idea to write:

$$A = \{3, 5, 7, \ldots\}$$

because we do not have enough information to decide whether A is the set of odd numbers larger than 1, or the set of prime numbers larger than 2.

When we use the constructive method, usually we have two parts in the description. The first part specifies which kind of objects we are

considering—the *universal set* (*e.g.*, integer numbers, such as cars produced during January 2005 in Detroit plants). The second part (if needed) follows the expression "such that" (usually represented by a vertical segment "|") and lists additional properties. So, the description of a set might look like:

$$A = \{n \in \mathbb{Z} \mid \text{the remainder of the division of } n \text{ by 2 is zero}\}$$
$$\text{B} = \{\text{cars produced during January 2005 in Detroit plants} \mid \text{they have four doors}\}$$

Usually, in a general setting the universal set is represented by U.

A set A is *contained* in a set B (or A is a *subset* of B) if every element of A is an element of B. In this case, we write $A \subseteq B$.

By definition of subset, every nonempty set has two trivial subsets, itself and $\emptyset$.

Two sets, A and B, are *equal* if the following two conditions are true:

1. $A \subseteq B$
2. $B \subseteq A$

The first of the two conditions states that every element of A is an element of B. The second states that every element of B is an element of A. Therefore, A and B have exactly the same elements.

Example 1. Let $A = \{n \in \mathbb{Z} \mid$ the remainder of the division of n by 2 is zero$\}$ and $B = \{$all integer multiples of 2$\}$. Prove that $A = B$.

Proof:

Part 1. $A \subseteq B$

Let x be a generic element of A (that is, x is any number satisfying the conditions to belong to the set A). We need to prove that x is an element of B as well.

As x is an element of A, we know that:

$$\frac{x}{2} = q + 0$$

where q is an integer number.

Thus, $x = 2q$. This means that x is a multiple of 2; therefore, x is an element of B.

Part 2. $B \subseteq A$

Let x be an element of B. We need to prove that x is an element of A as well.

Because x is in B, it is a multiple of 2. Therefore, $x = 2t$ with t integer number, and

$$\frac{x}{2} = \frac{2t}{2} = t.$$

Because the remainder of the division of x by 2 is zero, then x is an element of A.

Using both parts of this proof, we can conclude that $A = B$. ■

In some cases, it is easier to compare sets after making their descriptions as explicit as possible.

Example 2. Let $A = \left\{x \in \mathbb{R} \middle| \left|\frac{x}{2} - 1\right| < 5\right\}$ and $B = \{x \in \mathbb{R} | \ x$ is a number between the roots of the equation $x^2 - 4x - 96 = 0\}$. Prove that the two sets are equal.

Proof: We will simplify the descriptions of the two sets.

By definition of absolute value, the inequality:

$$\left|\frac{x}{2} - 1\right| < 5$$

is equivalent to the inequalities:

$$-5 < \frac{x}{2} - 1 < 5.$$

Adding 1 to all three parts of the preceding inequalities, we obtain:

$$-4 < \frac{x}{2} < 6$$

which is equivalent to:

$$-8 < x < 12.$$

Thus, we can rewrite:

$$A = \{x \in \mathbb{R} | -8 < x < 12\}.$$

The solutions of the equation $x^2 - 4x - 96 = 0$ are the numbers -8 and 12 (check this claim). Therefore,

$$B = \{x \in \mathbb{R} | -8 < x < 12\}.$$

At this point, it is evident that the two sets are equal. ■

To construct more interesting examples, we will consider two operations between sets.

Given two sets, A and B, their *union* is the set represented by $A \cup B$, and defined as:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

Given two sets, A and B, their *intersection* is the set represented by $A \cap B$, and defined as:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

"Venn diagrams" (John Venn [1834–1923] was a British logician) are commonly used to illustrate properties and operations between sets. Usually sets are represented by discs, which are labeled and placed inside a bigger rectangle that represents the universal set U.

Because a Venn diagram is a visual example, it is a good idea to try to represent a general situation and have sets overlapping each other, as shown in Diagram 1:

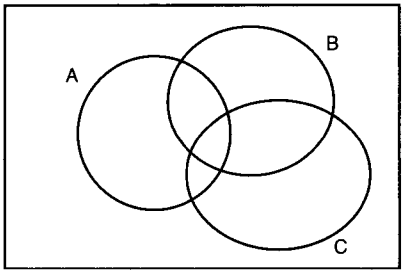


Diagram 1

The shaded area in Diagram 2 represents the set $A \cup B$:

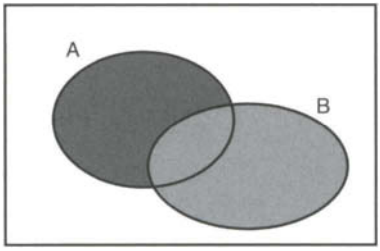


Diagram 2

The shaded area in Diagram 3 represents the set $A \cap B$:

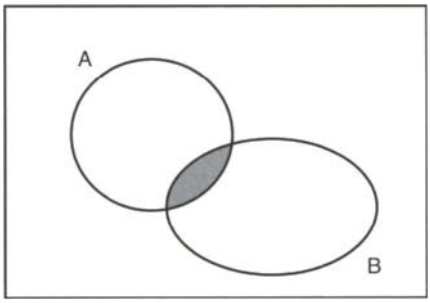


Diagram 3

Example 3. If A, B, and C are any three sets, then:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

(This is known as the distributive property of the intersection with respect to the union.)

Proof:

Part 1. $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
Let $x \in A \cap (B \cup C)$. We want to prove that $x \in (A \cap B) \cup (A \cap C)$.
Because $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in (B \cup C)$.
So, $x \in A$ and either $x \in B$ or $x \in C$.
Because we know that $x \in A$, we can consider two cases: either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$.
Thus, either $x \in (A \cap B)$ or $x \in (A \cap C)$.
Therefore, we can conclude that $x \in (A \cap B) \cup (A \cap C)$.

Part 2. $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$
Let $x \in (A \cap B) \cup (A \cap C)$. We want to prove that $x \in A \cap (B \cup C)$.
Because $x \in (A \cap B) \cup (A \cap C)$, then either $x \in (A \cap B)$ or $x \in (A \cap C)$.
So, either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$.
Therefore, in any case $x \in A$ and either $x \in B$ or $x \in C$.
Thus, $x \in A$ and $x \in (B \cup C)$.
So, we can conclude that $x \in A \cap (B \cup C)$.
By the conclusions proved in the two preceding parts, we can state that:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \qquad \blacksquare$$

Let's check the equality $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ using Venn diagrams.

We will start by constructing the set $A \cap (B \cup C)$, represented by the darkest shaded part in Diagram 4:

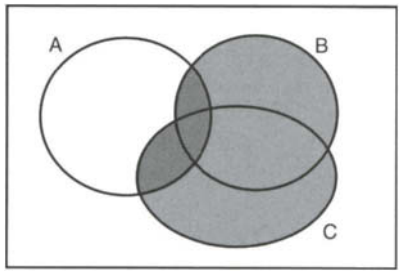


Diagram 4

We now consider the representation of the set $(A \cap B) \cup (A \cap C)$ shown in Diagram 5:

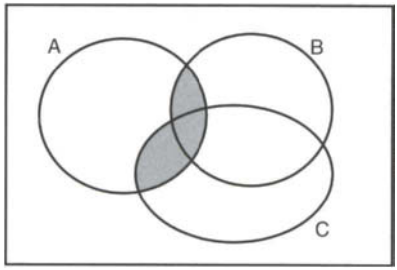


Diagram 5

Therefore, the two sets obtained using the Venn diagrams are equal.

The use of Venn diagrams does not provide a proof, but it offers a good illustration. Venn diagrams have the same role of examples. Moreover they become difficult to work with when the number of sets depicted becomes larger—for example, when dealing with arbitrary collections of four or more sets.

Example 4. Let $A = \{ x \in \mathbb{Z} \mid x \text{ is a multiple of } 5\}$ and $B = \{ x \in \mathbb{Z} \mid x \text{ is a multiple of } 7\}$. Then,

$$A \cap B = \{x \in \mathbb{Z} \mid x \text{ is a multiple of } 35\}.$$

Proof:

Part 1. $A \cap B \subseteq \{x \in \mathbb{Z} \mid x \text{ is a multiple of } 35\}$.

Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. This implies that x is a multiple of 5 and it is a multiple of 7. Therefore, $x = 5n$ and $x = 7m$, where n and m are integer numbers.

If we combine these two equalities, we obtain $5n = 7m$.

As 5 and 7 are prime numbers, $5n$ is divisible by 7 only if n is divisible by 7. Thus, $n = 7k$ for some integer number k.

Therefore, $x = 5n = 5(7k) = 35k$ for some integer number k. This means that x is a multiple of 35.

Part 2. $\{x \in \mathbb{Z} \mid x \text{ is a multiple of } 35\} \subseteq A \cap B$.

Let x be a multiple of 35. Therefore, $x = 35t$ for some integer number t.

Thus, x is divisible by 5 (so $x \in A$) and it is divisible by 7 (so $x \in B$).

This implies that $x \in A \cap B$.

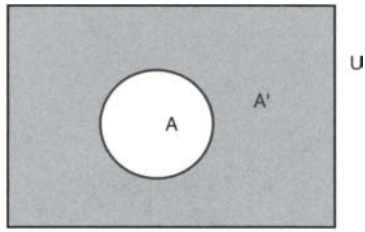
Therefore, the two sets are equal. ■

We will consider another set. The *complement* of a set A is the set of all elements that belong to the universal set U, but do not belong to the set A.

The *complement* of the set A can be denoted by a variety of symbols. The most commonly used are A', $C(A)$, and $\overline{A}$. We will use A'. Therefore,

$$A' = \{x \in U | x \notin A\}.$$

The part shaded in the following diagram represents the complement of the set A:



Example 5. Let $A \subset U$ and $B \subset U$. Then,

$$(A \cap B)' = A' \cup B'.$$

(This is known as one of De Morgan's laws. The proof of the other law—namely, $(A \cup B)' = A' \cap B'$—is left as an exercise. August De Morgan (1806–1871) was one of the first mathematicians to use letters and symbols in abstract mathematics.)

Proof:

Part 1. $(A \cap B)' \subseteq A' \cup B'$.

Let $x \in (A \cap B)'$. By definition of complement of a set this implies that $x \notin (A \cap B)$. Therefore, either $x \notin A$ or $x \notin B$. (Indeed, if x was an element of

both A and B, then it would be an element of their intersection, but we cannot exclude that x belongs to one of the two sets.)

Thus, by the definition of the complement of a set, either $x \in A'$ or $x \in B'$. This implies that $x \in A' \cup B'$.

Part 2. $A' \cup B' \subseteq (A \cap B)'$.

Let $x \in A' \cup B'$. Then either $x \in A'$ or $x \in B'$; that is, by the definition of complement of a set, either $x \notin A$ or $x \notin B$. This implies that x is not a common element of A and B; that is, $x \notin (A \cap B)$. Thus, we can conclude that $x \in (A \cap B)'$.

As both inclusions are true, the two sets are equal. ■

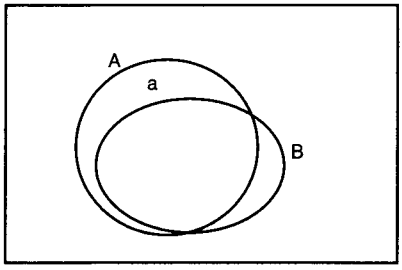
Sometimes the two inclusions can be proved at the same time. We could have proved the statement in Example 5 as follows:

$x \in (A \cap B)'$ if and only if $x \notin (A \cap B)$ if and only if $x \notin A$ or $x \notin B$

if and only if $x \in A'$ or $x \in B'$ if and only if $x \in A' \cup B'$.

While this kind of proof is clearly shorter than the one presented in Example 5, it can be trickier because there are fewer separate steps and it is less explicit. Therefore, proofs of this type can be more difficult to analyze and it becomes easier to overlook important details and make mistakes. (See Theorem 9 in the section Collection of Proofs section)

In order to prove that two sets, A and B, are *not* equal, it is sufficient to prove that at least one of the two inequalities ($A \subseteq B$ or $B \subseteq A$) does not hold. This means that it is enough to show that there is at least one element in one set that does not belong to the other.



Example 6. Let $A = \{\text{all odd counting numbers larger than 2}\}$ and $B = \{\text{all prime numbers larger than 2}\}$. These two sets are not equal.

Proof: We have already seen that all prime numbers larger than 2 are odd. Therefore, $B \subseteq A$.

Are all odd numbers larger than 2 prime numbers? The answer is negative, because the number 9 is odd, but it is not prime. So, $9 \in A$ but $9 \notin B$. Therefore, $A \subsetneq B$.

Thus, the two sets are not equal. ■

Example 7. Let $C = \{\text{all continuous functions on the interval } [-1, 1]\}$ and $D = \{\text{all differentiable functions on the interval } [-1, 1]\}$. These two sets are not equal.

Proof: All differentiable functions are continuous (a calculus book might be helpful for checking this claim), but not all continuous functions are differentiable.

Consider the function $f(x) = |x|$. This is continuous, but it is not differentiable at $x = 0$. Thus, $f \in C$ but $f \notin D$. Therefore, $C \subsetneq D$, and the two sets are not equal. ■

It is possible to define yet another operation between sets. Let A and B be two subsets of the same universal set U. The *difference* set is the set:

$$A - B = \{a \in A | a \notin B\}.$$

Example 8. Let A and B be two subsets of the same universal set U. The following equality holds true:

$$A - B = A \cap B'.$$

Proof: Let $x \in A - B$. By definition this will happen if and only if $x \in A$ and $x \notin B$. This is equivalent to stating that $x \in A$ and $x \in B'$. This will be true if and only if $x \in A \cap B'$.

EXERCISES

Prove the following statements.

1. For any three sets A, B, and C the following equality holds:

$$(A \cup B) \cup (A \cup C) = A \cup (B \cup C).$$

2. The sets $A = \{\text{all integer multiples of 2 and 3}\}$ and $B = \{\text{all integer multiples of 6}\}$ are equal.

3. Prove the second of De Morgan's laws:
$$(A \cup B)' = A' \cap B'.$$
4. For any three sets A, B, and C, the following equality holds:
$$(A \cap B) \cap C = A \cap (B \cap C)$$
(associative property of intersection)
5. Prove or disprove the following statement:

The sets $A = \{\text{all integer multiples of 16 and 36}\}$ and $B = \{\text{all integer multiples of 576}\}$ are equal.

6. Prove or disprove the following equalities, where A, B, and C are subsets of universal set U:

a. $A \cup (B \cap C) = (A \cup B) \cap C$;
b. $(A \cap B \cap C)' = A' \cup B' \cup C'$.

7. The sets:
$$A = \{(x, y) | y = x^2 - 1 \text{ with } x \in \mathbb{R} \text{ and } x \in \mathbb{R}\}$$
and
$$B = \left\{(x, y) | y = \frac{x^4 - 1}{x^2 + 1} \text{ with } x \in \mathbb{R} \text{ and } y \in \mathbb{R}\right\}$$
are equal.
8. Prove by induction that $(A_1 \cap A_2 \cap \ldots \cap A_n)' = A_1' \cup A_2' \cup \ldots \cup A_n'$ for all $n \geq 2$.
(See Example 5 for the base case $n = 2$.)
9. Prove by induction that, if a set has n elements, then it has 2^n subsets, where $n \geq 0$.

Fill in the details in the following proof.

10. A set $A \subseteq \mathbb{R}$ is *convex* if, whenever x and y are elements of A, the number $tx + (1 - t)y$ is an element of A for all values of t with $0 \leq t \leq 1$. The set $\{z | z = tx + (1 - t)y \text{ for } 0 \leq t \leq 1\}$ is called the *line segment* joining x and y.
Empty sets (sets with zero elements) and sets with one element are assumed to be convex.
Given this information, outline the proof of the following statement:
The intersection of two or more convex sets is a convex set.

Proof: Use mathematical induction on the number of sets.
Let A_1 and A_2 be two convex sets. If $A_1 \cap A_2$ is either empty or contains one element, then it is convex.
Let us assume that $A_1 \cap A_2$ has at least two distinct elements, x and y. Then x and y are elements of both A_1 and A_2. Because A_1 and A_2 are convex, the line segment joining x and y is contained in both sets A_1 and A_2. Therefore, it is contained in their intersection, $A_1 \cap A_2$.
Assume that if $A_1, A_2, \ldots, A_n$ are convex sets, then $A_1 \cap A_2 \cap \ldots \cap A_n$ is a convex set.
Prove that if $A_1, A_2, \ldots, A_n, A_{n+1}$ are convex sets, then $A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}$ is a convex set.
We can use the associative property of intersection (See Exercise 4) to write:
$A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1} = (A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}$.
The set $A_1 \cap A_2 \cap \ldots \cap A_n$ is convex by inductive hypothesis.
So $A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}$ is convex because it is the intersection of two convex sets.

EQUALITY OF NUMBERS

There are at least three common ways to prove that two numbers, call them a and b, are equal. We can do so by showing that:

1. The two inequalities $a \leq b$ and $b \leq a$ hold.
2. The equality $a - b = 0$ holds.
3. The equality $a/b = 1$ holds (in this case we need to be sure that $b \neq 0$).

It is preferable to use the second and third ways when we can set up algebraic expressions involving a and b. The first way is more useful when we have to compare numbers through the examination of their definitions and properties.

Example 1. If a and b are two positive integers, then their least common multiple is equal to the quotient between ab and the greatest common divisor of a and b; that is,

$$lcm(a, b) = \frac{ab}{GCD(a, b)}.$$

Proof: Let $d = GCD(a, b)$ and $L = lcm(a, b)$. We want to prove that:

$$L = \frac{ab}{d}.$$

(For definitions of the least common multiple and the greatest common divisor and their properties, see the front material of the book.)

By definition of $GCD(a, b)$, we can write:

$$a = dn \text{ and } b = dp$$

with n and p positive, relatively prime integers. Therefore,

$$\frac{ab}{d} = \frac{(dn)(dp)}{d} = dnp.$$

Let $M = dpn$. We want to prove that $M = L$.

Part 1. We will prove that $L \leq M$.

Clearly, M is a multiple of both a and b. Indeed, $M = pa$ and $M = nb$, with n and p positive integers. As M is a common multiple, it will be larger than (or equal to) the *least* common multiple, L.

Thus, $L \leq M$.

Part 2. We will prove that $M \leq L$.

By definition, L is a multiple of both a and b. Thus,

$$L = at \quad \text{and}$$
$$L = bs$$

where t and s are positive integers. Therefore, $at = bs$. Substituting a and b, we obtain:

$$L = dnt = dps.$$

Thus,

$$nt = ps.$$

This implies that n divides ps. As p and n are relatively prime, it follows that n divides s. Thus, $s = nk$ for some integer $k \geq 1$. So,

$$nt = pnk,$$

that is,

$$t = pk$$

for some integer $k \geq 1$. Then,

$$L = at = (dn)(pk) = dnpk.$$

This implies that $L = Mk$, with $k \geq 1$.

Thus, $L \geq M$.

Because $L \leq M$ and $L \geq M$, we can conclude that $L = M$. ■

Example 2. Let $f(x) = \frac{x}{x^2+1}$, and let y and z be two real numbers larger than 1. If $f(y) = f(z)$, then $y = z$.

(This proves that the function f is one-to-one on the interval $(1, +\infty)$. See the front material of the book for the definition of one-to-one.)

Proof: Because $f(y) = f(z)$, it follows that

$$\frac{y}{y^2+1} = \frac{z}{z^2+1}.$$

We can now multiply both sides of the equation by $(y^2 + 1)(z^2 + 1)$. This is a nonzero expression because $y^2 + 1 \neq 0$ and $z^2 + 1 \neq 0$. Therefore, we obtain:

$$zy^2 + z = z^2y + y$$

which can be simplified as:

$$(z - y)(1 - yz) = 0$$

Thus, either $z - y = 0$ or $1 - yz = 0$.

The first equality implies that $y = z$.

The second equality implies that $yz = 1$. This is not possible because y and z are two real numbers larger than 1. Therefore, the only possible conclusion is $y = z$. ■

There are at least two special ways to prove that a number is equal to zero, and both of them use the absolute value function and its properties. (See front material of the book for the definition of absolute value.)

Method 1. To prove that $a = 0$, we can prove that $|a| = 0$.

(This is true because, by definition of absolute value of a number, $|a| = 0$ if and only if $a = 0$.)

Method 2. Let a be a real number. Then $a = 0$ if and only if $|a| < \varepsilon$ for *every* real number $\varepsilon > 0$.

The second method is often used in calculus and analysis. We can prove that the two methods are equivalent.

Example 3. Let a be a real number. Then $|a| = 0$ if and only if $|a| < \varepsilon$ for every real number $\varepsilon > 0$.

Proof: Because this is an equivalence statement, the proof has two parts.

Part 1. If $|a| = 0$, then $|a| < \varepsilon$ for every real number $\varepsilon > 0$.

This implication is trivially true. Indeed, if $|a| = 0$, then $|a|$ is smaller than any positive number.

Part 2. If $|a| < \varepsilon$ for every real number $\varepsilon > 0$, then $|a| = 0$.

We will prove this statement by using its contrapositive.

Let us assume that $|a| \neq 0$. Does this imply that there exists at least one positive real number ε_0 such that $|a|$ is not smaller that ε_0?

Consider the number $\varepsilon_0 = |a|/2$. Then $0 < \varepsilon_0 < |a|$.

Because the contrapositive of the original statement is true, the original statement is true as well. ■

EXERCISES

Prove the following statements:

1. Let x and y be two real numbers.
   Then $(x - y)^5 + (x - y)^3 = 0$ if and only if $x = y$.
2. Let x and y be two real numbers. The two sequences $\{x^n\}_{n=2}^{\infty}$ and $\{y^n\}_{n=2}^{\infty}$ are equal if and only if $x = y$.
3. Let a, b, and c be three counting numbers. If a divides b, b divides c, and c divides a, then $a = b = c$.
4. Let a, b, and c be three counting numbers. Then $GCD(ac, bc) = c\,GCD(a, b)$.
5. Let a and b be two relatively prime integers. If there exists an m such that $(a/b)^m$ is an integer, then $b = 1$.

COMPOSITE STATEMENTS

The hypothesis and conclusion of a theorem might be composite statements. Because of the more complicated structure of this kind of statement, we have to pay very close attention. After analyzing a composite statement, we can check if it is possible to break it down into simpler parts, which can then be proved by using any of the principles and techniques already seen. Other times we will replace the original statement with another logically equivalent to it, but easier to handle.

Multiple Hypotheses

Multiple hypotheses statements are statements for which the hypotheses are composite statements, such as "If **A** and **B**, then **C**" and "If **A** or **B**, then **C**."

Let us start by examining statements of the form "If **A** and **B**, then **C**."

Proving that such a statement is true does not require any special technique, and some of these statements have already been included in previous sections. The main characteristic of this kind of statement is that the composite statement "**A** and **B**" contains several pieces of information, and we need to make sure that we use all of them during construction of the proof. If we do not, we are proving a statement different from the original. Always remember to consider possible implicit hypotheses.

Example 1. If b is a multiple of 2 and of 5, then b is a multiple of 10.

Proof:

Hypothesis:

A. The number b is a multiple of 2.
B. The number b is a multiple of 5.

(*Implicit hypothesis:* All the properties and operations of integer numbers can be used.)

Conclusion:

C. The number b is a multiple of 10.

By hypothesis **A**, the number b is a multiple of 2. So, $b = 2n$ for some integer n. The other hypothesis, **B**, states that b is a multiple of 5. Therefore, $b = 5k$ for some integer k. Thus,

$$2n = 5k.$$

Because $2n$ is divisible by 5, and 2 is not divisible by 5, we conclude that n is divisible by 5. Thus, $n = 5t$ for some integer number t. This implies that:

$$b = 2n = 2(5t) = 10t$$

for some integer number t. Therefore, the number b is a multiple of 10. ■

The proof of a statement of the form "If **A** and **B**, then **C**" can be constructed using its contrapositive, which is "If 'not **C**,' then *either* 'not **A**' *or* 'not **B**.'" (You might want to review the truth tables for constructing the negation of a composite statement introduced in the How To Construct the Negation of a Statement section.) This is a statement with multiple conclusions which is part of the next topic presented.

Let us construct the proof for Example 1 using the contrapositive of the original statement, just to start becoming more familiar with this kind of statement:

> "If the number b is not a multiple of 10, then either b is not a multiple of 2 or b is not a multiple of 5".

Proof: The two parts of the conclusion are "b is not a multiple of 2" and "b is not a multiple of 5." To prove that the conclusion is true, it is enough to prove that at least one of the two parts is true. (Keep reading for more details regarding this kind of statement.)

Assume that the number b is not a multiple of 10. Then, by the division algorithm,

$$b = 10q + r$$

where q and r are integers and $1 \leq r \leq 9$.

If r is an even number (*i.e.*, 2, 4, 6, 8), then we can write $r = 2t$, with t positive integer, and $1 \leq t \leq 4$. So,

$$b = 10q + 2t = 2(5q + t).$$

The number $5q + t$ is an integer, so the number b is divisible by 2. But b is not divisible by 5 because r is not divisible by 5.

Thus, in this case the conclusion is true because its second part is true.

If r is an odd number (*i.e.*, 3, 5, 7, 9), then b is not divisible by 2. In this case, the conclusion is true as well because its first part is true. ■

We will now consider statements of the form "If **A** or **B**, then **C**."

In this kind of statement, we know that the hypothesis "**A** or **B**" is true. This can possibly mean that:

1. Part **A** of the statement is true,
2. Part **B** of the statement is true,
3. Both parts **A** and **B** are true.

Because we do not know which one of the three cases to consider, we must examine all of them. It is important to notice that it is sufficient to concentrate on the first two cases, because the third case is a stronger case that combines the first two. Therefore, the proof of a statement of the form "If **A** or **B**, then **C**" has two parts (two cases):

1. Case 1. "If **A**, then **C**."
2. Case 2. "If **B**, then **C**."

Example 2. Let x, y, and z be counting numbers. If x is a multiple of z or y is a multiple of z, then their product xy is a multiple of z.

Proof:

Case 1. Let x be a multiple of z. Then $x = kz$ with k integer (positive because $x > 0$, $z > 0$). Therefore,

$$xy = (kz)y = (ky)z.$$

The number ky is a positive integer because k and y are positive integer. So xy is a multiple of z.

Case 2. Let y be a multiple of z. Then $y = nz$ with n integer (positive because $y > 0$, $z > 0$). Therefore,

$$xy = x(nz) = (xn)z.$$

The number xn is a positive integer because x and n are positive integer. So xy is a multiple of z. ■

The proof of a statement of the form "If **A** or **B**, then **C**" can be constructed using its contrapositive, which is "If 'not **C**,' then 'not **A**' and 'not **B**.'" (You might want to review the truth tables for constructing the negation of a composite statement introduced in the How To Construct the Negation of a Statement section.) This is again a statement with multiple conclusions which is part of the next topic.

The contrapositive of the original statement in Example 2 is the statement:

"Let x, y, and z be counting numbers. If the product xy is not a multiple of z, then x is not a multiple of z and y is not a multiple of z"

Multiple Conclusions

The most common kinds of multiple conclusion statements are:

1. If **A**, then **B** and **C**;
2. If **A**, then **B** or **C**.

We will consider these statements in some detail.

1. If **A**, then **B** and **C**.

The proof of this kind of statement has two parts:

i. If **A**, then **B**;
ii. If **A**, then **C**.

Indeed, we need to prove that each one of the possible conclusions is true, because we want all of them to hold. If we have already completed the proof that one of the two (or more) implications is true, we can use it to prove the remaining ones (if needed).

EXAMPLE 3. The lines $y=2x+1$ and $y=-3x+2$ are not perpendicular, and they intersect in exactly one point.

Proof:

Hypothesis:

A. The two lines have equations $y=2x+1$ and $y=-3x+2$.

(*Implicit hypothesis:* All the properties and relations between lines can be used.)

Conclusion:

B. The lines are not perpendicular.
C. The lines intersect in exactly one point.

Part 1. If **A**, then **B**.

Two lines are perpendicular if their slopes, m and m_1, satisfy the equation $m=-1/m_1$, unless one of them is horizontal and the other vertical, in which case one slope is equal to zero and the other is undefined.

The first line has slope 2 and the second has slope -3, so the lines are neither horizontal nor vertical. In addition, $-3\neq -1/2$. Thus, the lines are not perpendicular.

Part 2. If **A**, then **C**.

This second part is an existence and uniqueness statement: There is one and only one point belonging to both lines.

We can prove this part in two ways:

a. The given lines are distinct and nonparallel (as they have different slopes); therefore, they have only one point in common.
b. We can find the coordinates of the point(s) in common by solving the system:

$$\begin{cases} y = 2x + 1 \\ y = -3x + 2. \end{cases}$$

By substitution we have:

$$2x + 1 = -3x + 2.$$

The only solution of this equation is $x=1/5$.
The corresponding value of the y variable is $y=2(1/5)+1=7/5$.

Therefore, the lines have in common the point with coordinates (1/5, 7/5). This point is unique because its coordinates represent the only solution of the system formed by the equations of the two lines. ■

Example 4. If a number is even, then its second power is divisible by 4 and its sixth power is divisible by 64.

Proof:

Hypothesis:

A. The number n is even.

(*Implicit hypothesis:* All the properties and operations of integer numbers can be used.)

Conclusion:

B. The number n^2 is divisible by 4.
C. The number n^6 is divisible by 64.

Part 1. If **A**, then **B**.
By hypothesis the number n is even. Therefore, $n = 2t$ for some integer number t. This implies that:

$$n^2 = 4t^2.$$

As the number t^2 is an integer, it is true that n^2 is divisible by 4.

Part 2. If **A**, then **C**.
This implication can be proved in two ways.
First way: By hypothesis the number n is even. Therefore, $n = 2t$ for some integer number t. This implies that:

$$n^6 = 64t^6.$$

As the number t^6 is an integer, it is true that n^6 is divisible by 64.
Second way: We can use the result established in Part 1, as that part of the proof is indeed complete. When n is even, then $n^2 = 4k$ for some integer number k. Then we have:

$$n^6 = (n^2)^3 = (4k)^3 = 64k^3.$$

Because k^3 is an integer, it is true that n^6 is divisible by 64. ■

2. If **A**, then **B** or **C**.

In this case, we need to show that given **A**, then either **B** or **C** is true (not necessarily both). This means that we need to prove that at least one

of the possible conclusions is true; that is, if one of the two conclusions is false, then the other must be true. Thus, the best way to prove this kind of statement is to use the following one, which is logically equivalent to it:

If **A** and (not **B**), then **C**.

It might be useful to consider the truth tables for the two statements "If **A**, then **B** or **C**" and "If **A** and (not **B**), then **C**."

A	**B**	**C**	**B** or **C**	If **A**, then **B** or **C**
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

A	**B**	**C**	not **B**	**A** and (not **B**)	If **A** and (not **B**), then **C**
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

Similarly one can prove that the statements "If **A**, then **B** or **C**" and "If **A** and (not **C**), then **B**" are logically equivalent.

Example 5. Let n be a composite number larger than 1. Then n has at least one nontrivial factor smaller than or equal to $\sqrt{n}$.

Discussion: We have:

A. The number n is a composite number larger than 1.

Thus, $n = pq$ with $1 < p < n$ and $1 < q < n$.

(*Implicit hypothesis:* We can use all properties of counting and prime, non-prime numbers, divisibility, and the properties of square roots.)

B or C. Then either $p \leq \sqrt{n}$ or $q \leq \sqrt{n}$.

Proof: We will start by assuming that:

$$n = pq$$

where $1 < p < n$ and $1 < q < n$, and $p > \sqrt{n}$.

Multiplying this last inequality by q yields:

$$qp > q\sqrt{n},$$

that is,

$$n > q\sqrt{n}.$$

This implies $\sqrt{n} > q$. Thus, it is true that $\sqrt{n} \geq q$.

(Note that if $p = q$, then $p = q = \sqrt{n}$.) ■

The result stated in Example 5 is used to improve the speed of the search for possible prime factors of numbers.

Example 6. If x is a rational number and y is an irrational number, their sum, $x + y$, is an irrational number.

Discussion: We will prove the contrapositive of the given statement; that is, the statement "If the sum $x + y$ is a rational number, then either x is irrational or y is rational." Thus, let:

A. The sum $x + y$ is a rational number.

(*Implicit hypothesis:* As rational and irrational numbers are real numbers, we can use all the properties of real numbers and their operations.)

The fact that the numbers are called x and y is irrelevant. We can use any two symbols. We will keep using x and y to be consistent with the original statement.

B. The number x is irrational.

C. The number y is rational.

Therefore, we plan to prove the equivalent statement "If **A** and 'not **B**,' then **C**."

Proof: Assume that the number $x + y$ is rational and so is the number x.

Therefore, using the definition of rational numbers, we can write:

$$x + y = n/p$$

where $p \neq 0$, and n and p are integer numbers.

As x is rational, we can write $x = a/b$ with $b \neq 0$, where a and b are integer numbers. Thus, we have:

$$a/b + y = n/p$$

If we solve this equation for y, we obtain:

$$y = n/p - a/b = (nb - ap)/pb$$

where $pb \neq 0$ because $p \neq 0$ and $b \neq 0$.

The numbers $nb - ap$ and pb are integers because n, p, a, and b are integer numbers.

This information allows us to conclude that y is indeed a rational number.

As we have proved the contrapositive of the original statement to be true, the original statement is also true. ■

Example 7. Let a be an even number, with $|a| > 16$. Then either $a \geq 18$ or $a \leq -18$.

Discussion: In spite of its apparent simplicity, this statement has composite hypotheses and conclusions. Indeed, it is of the form "If **A** and **B**, then **C** or **D**," where:

A. The number a is even.

(*Implicit hypothesis:* We can use properties and operations of integer numbers.)
The fact that the number is called a is irrelevant.

B. $|a| > 16$.
C. $a \geq 18$.
D. $a \leq -18$.

Moreover, **B** is a composite statement. Indeed, **B** can be written as B_1 or B_2, with

$$\mathbf{B}_1 : a > 16 \quad \text{and}$$
$$\mathbf{B}_2 : a < -16.$$

Thus, the original statement can be rewritten as:

If (a is even and $a > 16$) or (a is even and $a < -16$),
then either $a \geq 18$ or $a \leq -18$.

The presence of an "or" in the hypothesis suggests the construction of a proof by cases.

Proof:
Case 1. We will prove the statement:

> If a is an even number and $a > 16$,
> then either $a \geq 18$ or $a \leq -18$.

As a is even and larger than 16, then it must be at least 18. Thus, $a \geq 18$, and the conclusion is true.
Case 2. We will prove the statement:

> If a is an even number and $a < -16$,
> then either $a \geq 18$ or $a \leq -18$.

As a is even and smaller than -16, then it cannot be -17, so it must be at most -18. Therefore, $a \leq -18$, and the conclusion is true. ■

EXERCISES

Prove the following statements.

1. If $x^2 = y^2$ where $x \geq 0$ and $y \geq 0$, then $x = y$.
2. If a function f is even and odd, then $f(x) = 0$ for all x in the domain of the function.
   (See the front material of the book for the definitions of even and odd functions.)
3. If n is a positive multiple of 3, then either n is odd or it is a multiple of 6.
4. If x and y are two real numbers such that $x^4 = y^4$, then either $x = y$ or $x = -y$.
5. Let A and B be two subsets of the same set U. Define the difference set $A - B$ as:

   $$A - B = \{a \in A | a \notin B\}.$$

   If $A - B$ is empty, then either A is empty or $A \subseteq B$.
6. Let A and B be two sets. If either $A = \emptyset$ or $A \subseteq B$, then $A \cup B = B$.
7. Fill in all the details and outline the following proof of the rational zero theorem:

   Let z be a rational zero of the polynomial:

   $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

which has all integer coefficients with $a_n \neq 0$ and $a_0 \neq 0$, where $n \geq 1$.
Let $z = p/q$ be written in its lowest terms, where $q \neq 0$.
Then q divides a_n and p divides a_0.

Proof: By hypothesis $P(z) = 0$. So,

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$

Therefore,

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n = 0.$$

Thus, we can solve for $a_n p^n$ and we obtain:

$$a_n p^n = -q(a_{n-1}p^{n-1} + \cdots + a_1 pq^{n-2} + a_0 q^{n-1})$$

which can be rewritten as $a_n p^n = -qt$, where $t = a_{n-1}p^{n-1} + \cdots + a_1 pq^{n-2} + a_0 q^{n-1}$ is an integer.
This implies that q divides $a_n p^n$. As p and q have no common factors, q divides a_n.
We can solve equation $(*)$ for $a_0 q^n$ to obtain:

$$a_0 q^n = -p(a_n p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}),$$

which can be rewritten as $a_0 q^n = -ps$, where $s = a_n p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}$ is an integer. Thus, p divides $a_0 q^n$.
Because p and q have no common factors, p must divide a_0. ■

LIMITS

The concepts of limits of functions and sequences are not easy ones to grasp, and their definitions have been the results of the mathematical and philosophical work of a number of mathematicians. This section includes only the most basic ideas.

The formal definition of the limit of a function at a real number c is usually stated as follows.

Definition. The real number L is said to be the *limit* of the function $f(x)$ at the point c, written as $\lim_{x \to c} f(x) = L$, if for *every* $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ and $x \neq c$, then $|f(x) - L| < \varepsilon$.

This statement is very modern (less than 200 years old) in its precise structure. For a long time mathematicians talked about vanishing

quantities and accepted and used more vague statements of the form: "The value of $f(x)$ gets as close as we want to $L$for all values of x sufficiently close to c." On one hand, the formal definition just presented removes the subjectivity of words such as "close" and "sufficiently close," while, on the other hand, for beginners in calculus and real analysis it seems to remove the intuitive meaning as well. But details are of paramount importance when trying to reach absolute precision. How can one make the transition from intuitiveness to rigor without missing the important points? Let us try to do so very carefully.

The statement "The value of $f(x)$ gets as close as we want to L" can be reworded as "The difference between the value of $f(x)$ and Lgets as small as we want"; that is, "$|f(x) - L|$ gets as small as we want." This means that the value of $|f(x) - L|$ has to be smaller than *every* positive number ε we can think of. So, the value of ε cannot be handpicked.

Because the value of $f(x)$ depends on x, $|f(x) - L|$ will be smaller than ε only for values of x suitably close to c. One question still remains: "How close to c should x be?" This is the same as asking "How small (large) does $|x - c|$ need to be?"

Therefore, to prove that $\lim_{x \to c} f(x) = L$, we need to prove the existence of an estimate for $|x - c|$ so that all the values of x that satisfy that estimate will make $|f(x) - L|$ smaller than the given ε. So, the proof of fact that the number L is indeed the correct limit is an existence proof. (See Existence Theorems section.)

Example 1. Prove that $\lim_{x \to 2}(3x - 5) = 1$.

Proof: According to the definition of limit, one must prove that for every given positive number ε there exists a positive number δ such that if x is a value that satisfies the requirement $|x - 2| < \delta$, then $|(3x - 5) - 1| < \varepsilon$.

The existence of δ will be proved explicitly, by giving a formula for it that depends on ε and possibly on 2, the value to which x is close.

Let us start by performing some algebraic steps, and then we will consider the geometric meaning of the result. The statement $|(3x - 5) - 1| < \varepsilon$ can be rewritten as:

$$|3x - 6| < \varepsilon$$

or

$$3|x - 2| < \varepsilon.$$

This inequality will hold true only when $|x - 2| < \frac{\varepsilon}{3}$.

Therefore, in this case we have $\delta = \varepsilon/3$, and because ε is positive, δ is positive as well.

Thus, when $|x-2|<\frac{\varepsilon}{3}$ it will follow that $|(3x-5)-1|<\varepsilon$. This proves that $\lim_{x\to 2}(3x-5)=1$. ■

The construction of the proof presented in Example 1 is very delicate, and it depends on the choice of the correct number as the limit. The algebraic steps performed cannot be replicated to show the existence of the number δ, if the limit was incorrectly chosen. We leave it to the reader to see how the proof breaks down if one tries to modify it to show that $\lim_{x\to 2}(3x-5)$ is some number other than 1.

Before proceeding further, let us look behind the algebraic approach and examine the geometric meaning of our findings. We will do so by choosing some values for ε and studying the detailed consequences of our choices. This will provide examples (which are useful but can never replace a proof), and it will also provide a visual approach to the concept of limit, which some people find quite helpful.

Consider $\varepsilon=4.5$. The conclusion just obtained states that $|(3x-5)-1|<4.5$ if we use values of x that satisfy the inequality $|x-2|<1.5$ (*i.e.*, values of x such that $-1.5<x-2<1.5$ or more explicitly $0.5<x<3.5$).

Consider the graph the function $f(x)=3x-5$ corresponding to the interval (0.5, 3.5) (see Figure 1.)

It can be observed that all the corresponding values of $f(x)$ fall between -3.5 and 5.5—that is, in less than 4.5 units from the value $L=1$, as shown in Figure 2.

If we now consider $\varepsilon=0.75$, we see that in order to have $|(3x-5)-1|<0.75$ we cannot use again all the values of x in the interval (0.5, 3.5). In this case, because we want the values of $f(x)$ to be closer than 0.75 to 1 (much closer than when we chose $\varepsilon=4.5$), we might need to choose x much closer to 2. Indeed, we will need $|x-2|<0.75/3=0.25$.

Figure 1

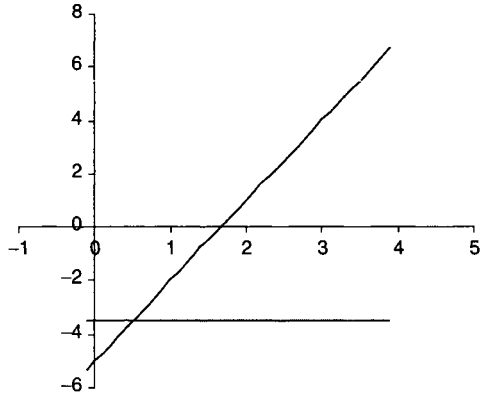
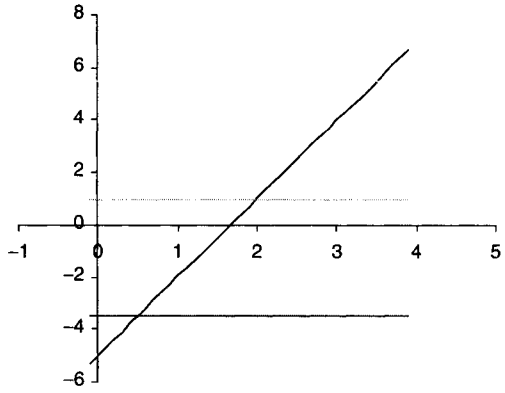


Figure 2



It is easy to check on the graph that when the value of x is in the interval (1.75, 2.25) the value of $f(x)$ will be closer than 0.75 to 1.

The definition of limit considered in this section assumes that both the number c and the limit L are finite, real numbers. Appropriate definitions can be stated to include $\pm\infty$ (*i.e.*, to use the extended real number system) and one-sided limits. This will not be done here, as it is beyond the goal of this presentation. The main goal of this section is to provide the reader with a first approach to the simpler examples of limits.

At this point, it might be useful to reread the definition of limit given above and note some important facts.

1. The number $\varepsilon > 0$ is assumed to be a given positive number, whose value cannot be specified, as the statement must be true for *all* $\varepsilon > 0$. This number can be very large or very small, and it provides the starting point for "finding" the number $\delta > 0$.
2. The number $\delta > 0$, whose existence needs to be proved, will depend on ε and usually on point c. In general, for a given ε the choice of the number δ is not unique (see Exercises 4 and 5 at the end of this section).
3. The function f might be undefined at c; that is, it might not be possible to calculate $f(c)$. Therefore, it is not always true that $L = f(c)$.
4. The variable x can approach the value c on the real number line from its left ($x < c$) and from its right ($x > c$); no direction is specified or can be chosen in the setting presented here.

Example 2. Prove that $\lim_{x \to 1} \dfrac{x-1}{x^2-1} = \dfrac{1}{2}$.

Proof: The goal is to prove that for every given positive number ε there exists a positive number δ such that if x satisfies the inequality $|x-1| < \delta$, then $\left|((x-1)/(x^2-1)) - (1/2)\right| < \varepsilon$.

The existence of δ will be proved explicitly, by providing a formula for it that depends on ε and possibly on 1.

Note that numbers -1 and 1 are not in the domain of the function $f(x) = (x-1)/(x^2-1)$. So, in particular, $f(x)$ is not defined at $c = 1$ (see Fact 3 above). While this might make the algebraic steps more delicate to handle, the logic steps will be similar to those in Example 1.

Let $\varepsilon > 0$ be given. How close to 1 will x have to be for the inequality $\left|((x-1)/(x^2-1)) - (1/2)\right| < \varepsilon$ to hold true? Let us start by simplifying the expression in the absolute value:

$$\left|\frac{x-1}{x^2-1} - \frac{1}{2}\right| = \left|\frac{1}{x+1} - \frac{1}{2}\right| = \left|\frac{2-(x+1)}{2(x+1)}\right|$$
$$= \left|\frac{1-x}{2(x+1)}\right| = \frac{|1-x|}{2|x+1|}.$$

By the properties of the absolute value function $|1-x| = |x-1|$. Thus,

$$\left|\frac{x-1}{x^2-1} - \frac{1}{2}\right| = |x-1|\frac{1}{2|x+1|}.$$

Once more, we need to prove that there exists a positive number δ such that, if $|x-1| < \delta$, then the quantity $\left|((x-1)/(x^2-1) - (1/2))\right| = |x-1|(1/(2|x+1|))$ will be smaller than the given $\varepsilon > 0$.

What is the largest value that the fraction $\dfrac{1}{2|x+1|}$ can have for x sufficiently close to 1?

Because the fraction is undefined for $x = -1$, we should avoid this value. Thus, let us choose an interval centered at 1 that does not include it, such as $-\dfrac{1}{4} < x < \dfrac{9}{4}$. This is an interval with center at 1 and radius $r = 5/4$ (*i.e.*, $|x-1| < 5/4$). (The choice of the radius is arbitrary; we can choose any positive number smaller than 2 to exclude $x = -1$.) For $r = 5/4$, we obtain the following estimates for the fraction $\dfrac{1}{2|x+1|}$.

$$-\frac{1}{4} < x < \frac{9}{4}$$
$$\frac{3}{4} < x+1 < \frac{13}{4}$$
$$\frac{3}{4} < |x+1| < \frac{13}{4}$$
$$\frac{3}{2} < 2|x+1| < \frac{13}{2}$$
$$\frac{2}{13} < \frac{1}{2|x+1|} < \frac{2}{3}.$$

Therefore, whenever $-(1/4) < x < (9/4)$,

$$\left|\frac{x-1}{x^2-1} - \frac{1}{2}\right| = |x-1|\frac{1}{2|x+1|} < |x-1|\frac{2}{3}$$

Then, the quantity $\left|\frac{x-1}{x^2-1} - \frac{1}{2}\right|$ will be smaller than ε for sure if $|x-1|\frac{2}{3} < \varepsilon$; that is, if $|x-1| < \frac{3}{2}\varepsilon$.

Let us choose $\delta = \text{minimum}\left\{\frac{5}{4}, \frac{3}{2}\varepsilon\right\}$. In this way, because $|x-1| < \delta \leq 5/4$ *and* $|x-1| < \delta \leq \frac{3}{2}\varepsilon$, we will have $\left|\frac{x-1}{x^2-1} - \frac{1}{2}\right| < \varepsilon$.

Therefore, we have proved that for every $\varepsilon > 0$ there exists a $\delta > 0$ (namely, $\delta = \text{minimum}\left\{\frac{5}{4}, \frac{3}{2}\varepsilon\right\}$), such that if $|x-1| < \delta$, then $\left|\frac{x-1}{x^2-1} - \frac{1}{2}\right| < \varepsilon$. This proves that $\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$. ■

It is possible to check whether the result obtained is indeed correct, and it should be emphasized that some mathematicians consider this checking process to be an important part of the proof.

In Example 2, let x satisfy the inequality $|x-1| < \delta$ with $\delta = \text{minimum}\{(5/4), (3/2)\varepsilon\}$ (so, in particular, $\delta \leq 5/4$ and $\delta \leq (3/2)\varepsilon$). Will it really follow that $\left|\frac{x-1}{x^2-1} - \frac{1}{2}\right| < \varepsilon$?

As seen before, $\left|\frac{x-1}{x^2-1} - \frac{1}{2}\right| = |x-1|\frac{1}{2|x+1|}$. Thus, we need to consider the two factors $|x-1|$ and $\frac{1}{2|x+1|}$. We already have an estimate for the first, as $|x-1| < \delta$. What about the second? Because $|x-1| < \delta < \frac{5}{4}$, it follows that:

$$\begin{aligned}
-\frac{5}{4} &< x-1 < \frac{5}{4} \\
-\frac{1}{4} &< x < \frac{9}{4} \\
\frac{3}{4} &< x+1 < \frac{13}{4} \\
\frac{3}{4} &< |x+1| < \frac{13}{4} \\
\frac{3}{2} &< 2|x+1| < \frac{13}{2} \\
\frac{2}{13} &< \frac{1}{2|x+1|} < \frac{2}{3}.
\end{aligned}$$

Thus,

$$\left|\frac{x-1}{x^2-1}-\frac{1}{2}\right| = |x-1|\frac{1}{2|x+1|} < \delta\frac{2}{3} < \left(\frac{3}{2}\varepsilon\right)\frac{2}{3} = \varepsilon.$$

It is considerably more difficult to prove that either $\lim_{x\to c} f(x) \neq L$ or that $\lim_{x\to c} f(x)$ does not exist. This is partly due to the fact that the definition of limit involves several quantifiers.

For example, one can state that $\lim_{x\to c} f(x) \neq L$ if there exists one $\varepsilon > 0$ for which there is no $\delta > 0$ such that if $|x-c| < \delta$, then $|f(x)-L| < \varepsilon$ for all x. In this case, L is assumed to be a real number.

This is equivalent to stating that $\lim_{x\to c} f(x) \neq L$ if there exists one $\varepsilon > 0$ such that for every $\delta > 0$ there is at least one x_δ with $|x_\delta - c| < \delta$ but such that $|f(x_\delta)-L| \geq \varepsilon$.

To prove that $\lim_{x\to c} f(x)$ does not exist, one would have to prove that $\lim_{x\to c} f(x) \neq L$ for all real numbers L

Example 3. Let $f(x) = \dfrac{x}{|x|}$. Then $\lim_{x\to 0} f(x)$ does not exist.

Proof: The function $f(x)$ is not defined at 0, but this does not necessarily imply the nonexistence of the limit.

(See Example 2. It might be useful to graph the function around 0 to study its behavior. The graph will show a "vertical jump" of 2 units at 0. Moreover, the values of $f(x)$ are positive for positive values of x and negative for negative values of x.)

Analytically, we need to prove that $\lim_{x\to 0} f(x) \neq L$ for all real numbers L. Therefore, we need to prove that it is possible to find an $\varepsilon > 0$ such that for every $\delta > 0$ there is at least one x_δ with $|x_\delta - 0| < \delta$ but such that $|f(x_\delta)-L| \geq \varepsilon$.

Consider $\varepsilon = 1/2$ (this choice is somewhat arbitrary; see the comments that follow this proof) and let δ be any positive number. As L is a real number, there are two possible cases: $L \leq 0$ and $L > 0$.

Case 1. $L \leq 0$. (We want to prove that the values of $f(x)$ are "not very close" to L. Because L is nonpositive, we can try using positive values of $f(x)$ that correspond to positive values of x.)

Let $x_\delta = \delta/4$, so $|x_\delta - 0| < \delta$. Note that $f(x_\delta) = \dfrac{\delta/4}{|\delta/4|} = \dfrac{\delta/4}{\delta/4} = 1$, and $|f(x_\delta)-L| = |1-L|$. Because $-L \geq 0, 1-L \geq 1$. Therefore,

$$|f(x_\delta)-L| = |1-L| = 1-L \geq 1 > \varepsilon.$$

Case 2. $L>0$. (We want to prove that the values of $f(x)$ are "not very close" to L. Because L is positive, we can try using negative values of $f(x)$ that correspond to negative values of x.)

Let $x_\delta = -\delta/4$, so that $|x_\delta - 0| < \delta$. Note that $f(x_\delta) = \dfrac{-\delta/4}{|-\delta/4|} = \dfrac{-\delta/4}{\delta/4} = -1$. Therefore,

$$|f(x_\delta) - L| = |-1 - L| = |-(1 + L)| = 1 + L > 1 > \varepsilon.$$

Thus, $\lim_{x\to 0} f(x)$ does not exist. ■

When proving that a limit does not exist, as done in Example 3, we need to find a value of ε so that, even for values of x close to c, $|f(x) - L| \geq \varepsilon$. There is no "recipe" for doing so. In general, the value of ε that will enable us to complete the proof depends on the behavior of the function around c, and smaller values of ε are more likely to work. In Example 3, we chose $\varepsilon = 1/2$, but, if one looks carefully through the steps of the proof, any value of ε smaller than or equal to 1 would be acceptable. Thus, in general the choice of ε is not unique.

The proof technique illustrated in Example 3 is not always easy to implement. Pragmatically, as one advances in the study of real analysis, one builds more and more tools to deal efficiently with the nonexistence of limits. Some of these tools rely on the structure of the real numbers (e.g., density properties of rational and irrational numbers) and on the relationships between functions and sequences. Therefore, while we will not examine these topics in depth, we think it is useful to consider at least the definition of limits of sequences. To use a self-contained approach, we will include the definition of sequence as well.

Definition. A *sequence* of real numbers is a function defined from the set of natural numbers $\mathbb{N}$ into the set of real numbers. The value of the function that corresponds to the number n is the real number usually indicated as a_n. The number a_n is referred to as a *term* of the sequence. Very often a sequence is identified with the ordered collection of real numbers $\{a_n\}_{n=1}^{\infty}$.

Definition. The real number L is said to be the *limit* of the sequence $\{a_n\}_{n=1}^{\infty}$, written as $\lim_{n\to\infty} a_n = L$, if for *every* $\varepsilon>0$ there exists a number N_ε such that $|a_n - L| < \varepsilon$ for all $n > N_\varepsilon$.

The definition is stating in a formal way the idea that as the number n gets "large enough," larger than a number N_ε that depends on ε, the corresponding term a_n gets "closer than ε" to the number L.

Example 4. Consider the sequence $\left\{a_n = \dfrac{2n-1}{n+1}\right\}_{n=1}^{\infty}$. Prove that $\lim\limits_{n\to\infty} \dfrac{2n-1}{n+1} = 2$.

Proof: Let $\varepsilon > 0$ be given. Is it possible to have $\left|\dfrac{2n-1}{n+1} - 2\right| < \varepsilon$ for n large enough?

(To get a feeling for the behavior of the terms of the given sequence, one might explicitly find some of them. Here is a very small collection of values: $a_1 = (2-1)/(1+1) = 0.5$, $a_2 = (4-1)/(2+1) = 1$, $a_3 = 1.25$, $a_7 = 1.625$, $a_{49} = 1.94$ $a_8 = 1.\overline{6}$, $a_{19} = 1.85$, and $a_{29} = 1.9$. So, it does seem to be true that as n gets larger the values of the terms of the sequence get closer and closer to 2.)

Again, let us start by simplifying the expression that involves n, as shown here:

$$\left|\frac{2n-1}{n+1} - 2\right| = \left|\frac{(2n-1) - 2(n+1)}{n+1}\right| = \left|\frac{-3}{n+1}\right| = \frac{|-3|}{|n+1|} = \frac{3}{n+1}.$$

Note that the number $n+1$ is positive, so it is equal to its absolute value. Thus, one must choose n large enough to have:

$$\frac{3}{n+1} < \varepsilon.$$

This is equivalent to requiring:

$$\frac{n+1}{3} > \frac{1}{\varepsilon}$$

or

$$n > \frac{3}{\varepsilon} - 1$$

Thus, let $N_\varepsilon = (3/\varepsilon) - 1$.

For all $n > N_\varepsilon = (3/\varepsilon) - 1$ one will have $\left|\dfrac{2n-1}{n+1} - 2\right| < \varepsilon$, therefore proving that 2 is indeed the limit of the sequence. ■

In the case of sequences as well, it is possible to check that the estimate found for N_ε is correct. Again, for some mathematicians this checking process is an important part of the proof.

In Example 4, let n be any number such that $n > N_\varepsilon = (3/\varepsilon) - 1$, and let $\varepsilon > 0$ be given. Is it true that $\left|\dfrac{2n-1}{n+1} - 2\right| < \varepsilon$? Let us check. Consider the equality:

$$\left|\frac{2n-1}{n+1} - 2\right| = \left|\frac{(2n-1) - 2(n+1)}{n+1}\right| = \frac{3}{n+1}$$

Because $n > (3/\varepsilon) - 1$, it follows that $n + 1 > (3/\varepsilon) > 0$. Therefore, $\frac{1}{n+1} < \frac{\varepsilon}{3}$. This inequality is equivalent to the inequality $\frac{3}{n+1} < \varepsilon$. So, we can conclude that $\left|\frac{2n-1}{n+1} - 2\right| < \varepsilon$ whenever $n > N_\varepsilon = (3/\varepsilon) - 1$.

One thing to be noticed in the definition of limit is that while ε must always be a positive number, the number N_ε can be negative, and, while n is always an integer number, N_ε does not need to be integer. Some authors do specify that N_ε should be a nonnegative integer, but there is no need to do so.

If N_ε happens to be negative, the statement $n > N_\varepsilon$ is always true. Thus, the inequality $|a_n - L| < \varepsilon$ will be true for all values of n.

If N_ε is not an integer, then n will just be any integer larger than it.

In Example 4, when $\varepsilon = 4$, $N_4 = -1/4$. This means that the statement $\left|\frac{2n-1}{n+1} - 2\right| < 4$ is always true. It is easy to check that this is indeed the case by looking at the values of the terms of the sequence $\left\{a_n = \frac{2n-1}{n+1}\right\}_{n=1}^{\infty}$.

When $\varepsilon = 4/5$, $N_{4/5} = 2.75$. This means that the statement $\left|\frac{2n-1}{n+1} - 2\right| < \frac{4}{5}$ is true for all numbers n larger than 2.75. As n is an integer, it means that it will be true for $n \geq 3$.

Example 5. Let $\left\{a_n = \frac{3n}{2n-1}\right\}_{n=1}^{\infty}$. Then $\lim_{n\to\infty} \frac{3n}{2n-1} = \frac{3}{2}$.

Proof: Let $\varepsilon > 0$ be given. Is it possible to have $\left|\frac{3n}{2n-1} - \frac{3}{2}\right| < \varepsilon$ for n large enough?

We will start by simplifying the expression in the absolute value:

$$\left|\frac{3n}{2n-1} - \frac{3}{2}\right| = \left|\frac{2 \times 3n - 3(2n-1)}{2(2n-1)}\right| = \left|\frac{3}{2(2n-1)}\right| = \frac{3}{2(2n-1)}.$$

Therefore, one needs to solve the inequality:

$$\frac{3}{2(2n-1)} < \varepsilon.$$

Doing so yields the result $n > \frac{1}{2}\left(\frac{3}{2\varepsilon} + 1\right)$. Thus, let $N_\varepsilon = \frac{1}{2}\left(\frac{3}{2\varepsilon} + 1\right)$. ■

In the case of limits of sequences, similarly to the case of limits of functions, given an $\varepsilon > 0$ the corresponding N_ε is not unique. See Exercises 9 and 10 at the end of this section.

For sequences, as is the case with functions as well, it is usually easier to prove the existence of a limit than its nonexistence. Indeed, to prove

that $\lim_{n\to\infty} a_n \neq L$, one needs to prove that there exists an $\varepsilon > 0$ such that for all numbers N there is an $n > N$ such that $|a_n - L| \geq \varepsilon$. In this statement, L is a real number.

To prove that $\lim_{n\to\infty} a_n$ does not exist, one needs to prove that the statement above is true for all real numbers; that is, $\lim_{n\to\infty} a_n \neq L$ for all real numbers L

EXAMPLE 6. Consider the sequence $\{a_n = (-1)^n\}_{n=1}^{\infty}$. Prove that its limit does not exist.

Proof: To show that the conclusion is true we need to prove that there exists an $\varepsilon > 0$ such that for all numbers N there is an $n > N$ such that $|a_n - L| \geq \varepsilon$, where L is any real number.

(As the values of the terms of the sequence oscillate between -1 and 1, it might make sense to use a value of ε smaller than 1.) Consider $\varepsilon = 9/10$. We have to consider two possible cases for L: $L \leq 0$ and $L > 0$.

Case 1. $L \leq 0$. (We want to prove that the terms of the sequence are somewhat "far" from L. As L is nonpositive, and some of the terms of the sequence are positive, we could try to use them to reach the goal.). Let N be any number and let t be an even number, where $t > N$. Then $a_t = (-1)^t = 1$. So, $|a_t - L| = |1 - L|$. Because $-L \geq 0$, $1 - L \geq 1$. Therefore,

$$|a_t - L| = |1 - L| = 1 - L \geq 1 > \varepsilon.$$

Case 2. $L > 0$. (We want to prove that the terms of the sequence are somewhat "far" from L Because L is positive and some of the terms of the sequence are negative, we could try to use them to reach the goal.). Let N be any number and let s be an odd number, where $s > N$. Then $a_s = (-1)^s = -1$. So,

$$|a_s - L| = |-1 - L| = |-(1 + L)| = 1 + L > 1 > \varepsilon.$$

Thus, the sequence does not have a limit. ■

EXERCISES

1. Prove that $\lim_{x\to 1} 3x^2 + 2 = 5$.
2. Prove that $\lim_{x\to 2} \dfrac{1}{x^2 + 1} = \dfrac{1}{5}$.

3. Prove that $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1} = \dfrac{3}{2}$.
4. The choice of δ is not unique. In Example 1, we proved that when $\varepsilon = 4.5$, we can use $\delta = 1.5$. Show that if one chooses $\delta = 0.9$, it is still true that $|(3x-5)-1| < 4.5$.
5. Prove that in Example 1 one would have $|(3x-5)-1| < 4.5$ for all values of x with $|x-2| < \delta$ for any $\delta \leq 1.5$.
6. Prove that $\lim_{n\to\infty} \dfrac{1}{3n+1} = 0$.
7. Prove that $\lim_{n\to\infty} \dfrac{1}{n^2+1} = 0$.
8. Prove that $\lim_{n\to\infty} \dfrac{5n+1}{3n-2} = \dfrac{5}{3}$.
9. In Example 4, we determined that when $\varepsilon = \frac{4}{5}$, $N_{4/5} = 2.75$. Prove that if we use $M_{4/5} = 16$, then $\left|\dfrac{2n-1}{n+1} - 2\right| < \dfrac{4}{5}$ for all $n > M_{4/5}$.
10. Consider the sequence $\left\{a_n = \dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$ whose limit is zero. Given $\varepsilon > 0$, prove that the statement $\left|\dfrac{n+1}{n^2} - 0\right| < \varepsilon$ is true for all n such that: (a) $n > N_\varepsilon = \dfrac{1}{2\varepsilon}\left(1 + \sqrt{1+4\varepsilon}\right)$; (b) $n > M_\varepsilon = \dfrac{1+\varepsilon}{\varepsilon}$.
(This exercise is meant to reinforce the fact that for a given ε one can have several choices for the number N.)

Review Exercises

Discuss the truth of the following statements. Prove the ones that are true; find a counterexample for each one of the false statements. Exercises with the symbol (*) require knowledge of calculus or linear algebra.

1. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points in the plane, then the distance between the two of them, defined as:

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

   is a positive number.

2. Let a be a real number. Then the opposite of a is unique.

3. If n is any positive integer number, then $\ln n < n$.
   Prove this statement in all of the following ways:
   (a) By induction.
   (b) By graphing the two functions $f(x) = \ln x$ and $g(x) = x$ and comparing them. Use only $x \geq 1$ as the statement is only about positive integers.
   (c) By studying the function $h(x) = \ln x/x$ for $x \geq 1$. (*)
   (Show that the function is bounded by 0 and 1.)
   (d) By studying the function $g(x) = \ln x - x$ for $x \geq 1$. (*)
   (Show that $g(x) < 0$ for all $x \geq 1$.)

4. (a) Are the following two sets equal?
   $A = \{$all integer numbers that are multiples of 15$\}$
   $B = \{$all integer numbers that are multiples of 3 and 5$\}$
   (b) Are the following two sets equal?
   $A = \{$all integer numbers that are multiples of 15$\}$
   $B = \{$all integer numbers that are either multiples of 3 or multiples of 5$\}$

5. Let a and b be two real numbers with $a \neq 0$. The solution of the equation $ax = b$ exists and is unique.

6. The counting number n is odd if and only if n^3 is odd.

7. Let a and b be two real numbers. The following statements are equivalent:
   (a) $a \leq b$ and $a \geq b$
   (b) $a - b = 0$.

8. Every nonzero real number has a unique reciprocal.

9. Let p, q, and n be three positive integers. If p and q have no common factors, then q does not divide p^n.

10. For every integer $n > 0$

$$\frac{1}{1}\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{4} + \cdots + \frac{1}{n}\frac{1}{n+1} = \frac{n}{n+1}.$$

11. $\sqrt{2}$ is an irrational number.

12. Prove algebraically that two distinct lines have at most one point in common.

13. All negative numbers have negative reciprocals. (See Exercise 8.)

14. Let $\left\{a_n = \dfrac{3n+2}{n}\right\}_{n=1}^{\infty}$. Then $\lim_{n\to\infty} a_n = 3$.

15. The remainder of the division of a polynomial $P(x)$ by the monomial $x - a$ is the number $P(a)$.

16. Let $P(x)$ be a polynomial of degree larger than or equal to 1. The following statements are equivalent:
   (1) The number $x = a$ is a root of $P(x)$.
   (2) The polynomial $P(x)$ can be exactly divided by the monomial $x - a$.
   (3) The monomial $x - a$ is a factor of the polynomial $P(x)$.

17. Let f be a differentiable function at the point $x = a$. Then f is continuous at that point. (*)

18. All prime numbers larger than two are odd.

19. Let A be a 2×2 matrix with real entries. The following statements are equivalent: (*)
    (1) The matrix A has an inverse.
    (2) The determinant of A is non equal to zero.
    (3) The system $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has only the trivial solution $x=0$, $y=0$.
20. Let $f(x) = 3x^2 + 7x$. Then $\lim_{x \to 1} f(x) = 10$.
21. For all positive integer numbers k:

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}.$$

22. Let a and b be two real number. If $ab = \frac{(a+b)^2}{4}$, then $a=b$.
23. Let a and b be two real number. If $ab = \frac{(a+b)^2}{2}$, then $a=b=0$.
24. For all integers $k \geq 2$,

$$\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{2k} > \frac{1}{2}.$$

25. Let a, b, and c be three integers. If a is a multiple of b and b is a multiple of c, then a is a multiple of c.
26. Let p be a nonzero real number. Then p is rational if and only if its reciprocal is a rational number.
27. Let $\{a_n = (-1/2)^n\}_{n=1}^{\infty}$. Then $\lim_{n \to \infty} a_n = 0$.
28. Let a, b, and c be three consecutive integers. Then 3 divides $a + b + c$.
29. Let k be a whole number. Then $k^3 - k$ is divisible by 3. How does this exercise relate to the previous exercise?
30. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers converging to the number L; that is $\lim_{n \to \infty} a_n = L$
    If $a_n > 0$ for all n, then $L \geq 0$.
31. Let $f(x) = \sqrt{x}$. Then $\lim_{x \to 3} f(x) = \sqrt{3}$.
32. If $ad - bc \neq 0$, then the system:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

has a unique solution. The numbers a, b, c, d, e, and f are all real numbers.

33. Let $k \geq 6$ be an integer. Then $2^k > (k+1)^2$.
34. There exists a number k such that $2^k > (k+1)^2$.
35. If t is a rational number and q is a rational number, then $t + q$ is an irrational number.
36. There are three consecutive integer numbers a, b, and c such that 3 divides $a + b + c$. (See Exercise 28.)
37. Let n be an integer. If n is a multiple of 5, then n^2 is a multiple of 125.
38. For every integer n the number $n^2 + n$ is always even.
39. Let $k \geq 6$ be an integer. Then $k! > k^3$.
40. Let $\left\{a_n = \frac{(-1)^n}{5}\right\}_{n=1}^{\infty}$. Then $\lim_{n\to\infty} a_n$ does not exist.

Exercises Without Solutions

Discuss the truth of the following statements. Prove the ones that are true; find a counterexample for each one of the false statements. Exercises with the symbol (*) require knowledge of calculus or linear algebra.

GENERAL PROBLEMS

1. (I) Write each of the following statements in the form "If **A**, then **B**". (II) Construct the contrapositive of each statement.
   (a) Every differentiable function is continuous.
   (b) The sum of two consecutive numbers is always an odd number.
   (c) The product of two consecutive numbers is always an even number.
   (d) No integer of the form $n^2 + 1$ is a multiple of 7.
   (e) Two parabolas having three points in common coincide.
   (f) Let b and c be any two real numbers with $b \leq c$, and let a be their arithmetic average, defined as $a = \frac{b+c}{2}$. Then $b \leq a \leq c$.
2. The number $\sqrt{7}$ is irrational.
3. The only prime of the form $n^5 - 1$ is 31.
4. There is a differentiable function whose graph passes through the three points $(-1, 0)$, $(0, -3)$, and $(1, 5)$. (*)

5. The reciprocal of a nonzero number of the kind $z = a + b\sqrt{5}$, with a and b real numbers, is a number of the same kind.
6. If x is a positive real number, then $x^3 \geq x$.
7. Let n be a natural number and x a fixed positive irrational number. Then $\sqrt[n]{x}$ is always irrational.
8. Let n be an odd number. Then $n(n^2 - 1)$ is divisible by 24.
9. Let f be a differentiable increasing function. Then $\sqrt[3]{f(x)}$ is an increasing function. (*)
10. Let a, b, c, and n be four positive integers. The numbers a, b, and c are divisible by n if and only if $a + b + c$ is divisible by n.
11. Consider the equation $ax^2 + bx + c = 0$, with $a \neq 0$, and $b^2 - 4ac \geq 0$. Then its solutions are $x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.
12. Let a and b be any two real numbers. Prove the following statements:
    (a) $(a + b)^2 \geq 4ab$
    (b) $(a + b)^2 = 4ab$ if and only if $a = b$.
13. The sum of two consecutive numbers is divisible by 2.
14. Let a and d be two *fixed* positive integer numbers. Then

$$a + (a + d) + (a + 2d) + (a + 3d) + ... + (a + nd) = \frac{(n + 1)(2a + nd)}{2}$$

    for all integers $n \geq 1$.
15. Let n be a natural number. Then n is a multiple of 7 if and only if n^3 is a multiple of 7.
16. Let $f(x) = 15x + 7$. Then $\lim_{x \to 2} f(x) = 37$.
17. The product of two consecutive numbers is divisible by 2.
18. If a, b, and c are three integers such that $a^2 + b^2 = c^2$ (*i.e.*, they are a Pythagorean triple), then they cannot all be odd.
19. The square of an odd integer is a number of the form $8t + 1$, where t is an integer.
20. Let n be a number that is not a multiple of 3. Then either $n + 1$ or $n - 1$ is a multiple of 3.
21. If $(a + b)^3 = a^3 + b^3$, then either $a = 0$ or $b = 0$.

22. Prove that if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for all $n \geq 2$. (*)
23. Let a and b be two integers such that b is a multiple of a. Then b^k is a multiple of a^k for every natural number k.
24. The squares of two consecutive integers are not consecutive integers.
25. Let x and y be two negative numbers. Show that the following four statements are equivalent:
    (a) $|x| < |y|$
    (b) $0 < \frac{x}{y} < 1$
    (c) $1 < \frac{y}{x}$
    (d) $x^2 < y^2$
26. An integer is a multiple of 5 if and only if its unit digit is either a 5 or a 0.
27. The squares of two consecutive positive integers are not consecutive positive integers.
28. Let f be a function defined for all real numbers. The function f is decreasing if and only if $\dfrac{f(c) - f(x)}{c - x} < 0$ whenever $x \neq c$.
29. The difference of two irrational numbers is always an irrational number.
30. Let $f(x) = \dfrac{x - 3}{x^2 - 9}$. Then $\lim_{x \to 3} f(x) = 1/6$.
31. If is an integer with $n \geq 1$, then $(n + 1)^2 \geq 2n^2$.
32. Let a be a positive number. Then $a < 1$ if and only if $a^3 < a$.
33. Let n be a nonnegative integer. The number $10^{2n+1} + 1$ is divisible by 11.
34. The set A of all odd integer multiples of 3 coincides with the set $B = \{n = 6k + 3$ where k is an integer$\}$.
35. Let f and g be two functions defined for all real numbers and such that the function $f \circ g$ is well-defined for all real numbers. If f is a differentiable function and g is a nondifferentiable function, then $f \circ g$ is nondifferentiable. (*)
36. Let f and g be two increasing functions defined for all real numbers. Then the function $h(x) = f(x)g(x)$ is an increasing function.
37. Consider the sequence $\left\{a_n = \dfrac{2n + 1}{3n + 5}\right\}_{n=1}^{\infty}$. Then $\lim_{n \to \infty} a_n = \dfrac{2}{3}$.
38. (a) If $a_1, a_2, \ldots, a_n$ is a finite collection of rational numbers, then the sum:

$$S = a_1 + a_2 + \ldots + a_n \quad \text{is a rational number.}$$

(b) Is the previous statement true if one considers an infinite sequence of rational numbers? That is: If $a_1, a_2, \ldots, a_n, \ldots$ is an infinite sequence of rational numbers, is the sum:

$$S = a_1 + a_2 + \ldots + a_n + \ldots\ldots = \sum_{k=1}^{\infty} a_k$$

a rational number? (*)

39. Let f, g, and h be three functions defined for all real numbers such that $f(x) \leq h(x) \leq g(x)$ for all x. If f and g are decreasing, then h is decreasing.

40. The equation $\sin x = -x + 1/2$ has a unique solution for $0 \leq x \leq \pi/2$.

41. The product of four consecutive integers increased by 1 is always a perfect square.
(*Hint:* Try the square of a trinomial.)

42. Let A and B be two nonempty subsets of the same set U. Then either $B \subseteq A'$ or $A \cap B \neq \emptyset$.

43. There exist integer numbers a, b, and c such that bc is a multiple of a, but neither b nor c is a multiple of a.

44. Let n be a natural number larger than 3. Then $2^n > n!$

45. Let f be a function defined for all real numbers. The function f is even if and only if its graph is symmetric with respect to the y-axis. (A graph is symmetric with respect to the y-axis if whenever the point (x, y) belongs to it, the point $(-x, y)$ will belong to it as well.)

46. The systems $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ and $\begin{cases} ax + by = e \\ (a-c)x + (b-d)y = e - f \end{cases}$ have the same solutions.
(*Hint:* Prove that (t, s) is a solution of the first system if and only if it is a solution of the second system.)

47. Let k be a natural number. An integer of the form $16k + 5$ is never a perfect square.

48. Let n be an integer. Then the following four statements are equivalent:
(a) n is odd.
(b) n^2 is odd.
(c) $(n-1)^2$ is even.
(d) $(n+1)^2$ is even.

49. Let f be a positive function defined for all real numbers and never equal to zero. Then the following statements are equivalent:
(a) f is an increasing function.

(b) The function g, defined as $g(x) = \frac{1}{f(x)}$, is decreasing.

(c) The function k_n, defined as $k_n(x) = nf(x)$, is increasing for all positive numbers n.

50. The number $3^n - 1$ is divisible by 2 for all natural numbers n.

51. Let n be a positive multiple of 3, with $n > 3$. Then either n is a multiple of 6 or it is a multiple of 9.

52. For every counting number n, $\sum_{i=1}^{n} i = \sqrt{\sum_{i=1}^{n} i^3}$.
(*Hint:* Start by squaring both sides of the equality.)

53. There is a differentiable function f such that $0 \leq f(x) \leq 1$ and $f(0) = 0$. (*)

54. If ab is divisible by 10, then either a or b is divisible by 10.

55. Let f be a nonconstant function. Then f cannot be even and odd at the same time.

56. Let a be a positive integer. If 3 does not divide a, then 3 divides $a^2 - 1$.

57. There exists a set of four consecutive integers such that the sum of the cubes of the first three is equal to the cube of the largest number.

58. Let A, B, and C be three subsets of the same set U. Then

$$A - (B \cap C) = (A - B) \cup (A - C).$$

59. A five-digit palindrome number is divisible by 11.

60. Let m and n be two integer numbers. Then the following statements are equivalent:
(a) m and n are both odd numbers.
(b) mn is an odd number.
(c) m^2n^2 is an odd number.

61. There exist irrational numbers a and b such that a^b is an integer.

62. Let $f(x) = \sqrt{x+1}$. Then $\lim_{x \to 3} f(x) = 2$.
(*Hint:* Use, in a suitable way, the conjugate of $\sqrt{x+1} - 2$.)

63. Let f, g, and h be three functions defined for all real numbers such that h and g are increasing and $h(x) \leq f(x) \leq g(x)$ for all x. Then f is an increasing function.

64. Let n be an integer number. The following statements are equivalent:
(a) n is divisible by 5.
(b) n^2 is divisible by 25.
(c) n^3 is divisible by 125.

65. Let $\left\{a_n = \frac{n^2-1}{n^2+1}\right\}_{n=1}^{\infty}$. Then $\lim_{n\to\infty} a_n = 1$.
66. Let c be a perfect square (*i.e.*, $c = a^2$ for some integer number a). Then the number of distinct divisors of c is odd.
67. Let a, b, c, and d be natural numbers such that b is a multiple of a and d is a multiple of c. Then bd is a multiple of ac.
68. Let $A_1, A_2, \ldots, A_n$ be any n sets. Then $(A_1 \cup A_2 \cup \ldots \cup A_n)' = A_1' \cap A_2' \cap \ldots \cap A_n'$ for all $n \geq 2$.
69. There exists a third-degree polynomial whose graph passes through the points $(0, 1)$, $(-1, 3)$, and $(1, 3)$.
70. The Fibonacci sequence $f_1, f_2, \ldots, f_n$ is defined recursively as $f_1 = 1, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$. Then for $n > 2$:
    (a) $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$
    (b) $f_n > \left(\frac{1+\sqrt{5}}{2}\right)^n$
    (*Hint:* Write several terms of the sequence to study its behavior.)
71. There exists a positive integer n such that $n! < 3^n$.
72. Let a and b be two real numbers with $a < b$. Then there exists a unique number c, with $a < c < b$ such that $|a - c| = \frac{|b-c|}{3}$.
73. Let $f(x) = |x|$. Then $\lim_{x\to 0} f(x) = 0$.
74. Let a and b be two rational numbers, with $a < b$. Then there exist at least four rational numbers between a and b.
75. The sum of two increasing functions is an increasing function.
76. The sum of two prime numbers is not a prime number.
77. There is a digit that appears infinitely often in the decimal expansion of $\sqrt{2}$.
78. Let A and B be subsets of a universal set U. The *symmetric difference* of A and B, indicated as $A \oplus B$, is the subset of U defined as follows:

$$A \oplus B = (A - B) \cup (B - A)$$

Show $A \oplus B$ on a Venn diagram.
Prove that $(A - B) \cap (B - A) = \emptyset$.
Prove that $A \oplus B = (A \cup B) - (A \cap B)$.
(You can use a Venn diagram as an example, but you need to write a general proof as well.)
Prove that $A \oplus B = B \oplus A$.

(This show that this operation is a symmetric (commutative), because the roles of A and B can be changed without changing the final result.)

79. Let A, B, C, and D be any subsets of a universal set U. In each case, either give a proof of the fact that the equality is true or find a counterexample to show that it is false. Do not use Venn diagrams to prove the truth of equalities.

$$(A - B) \cap C = A \cap (B' \cap C)$$

$$(A \cup B \cup C)' = A' \cap B' \cap C'$$

$$(A \cup B) \cup (C \cap D) = (A \cup B \cup C) \cap D$$

80. Let $\left\{a_n = \frac{1}{n^3+1}\right\}_{n=1}^{\infty}$. Then $\lim_{n\to\infty} a_n = 0$.

81. Let $P(x)$ and $Q(x)$ be two polynomials such that $P(x) = (x^2 + 1)Q(x)$. The solution sets of the two polynomials coincide.
(*Hint:* Prove that x_0 is a zero of $P(x)$ if and only if it is a zero of $Q(x)$.)

82. Let ℓ_1 and ℓ_2 be two nonhorizontal distinct lines perpendicular to a third line ℓ_3. Then ℓ_1 and ℓ_2 are parallel to each other.
(*Hint:* work on the slopes of the three lines.)

83. There exists a unique prime of the form $n^3 - 1$.

84. Let x be a real number. If x^2 is not a rational number, then x is not a rational number.

85. The sum of two odd functions is an odd function.

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Collection of Proofs

The following collection includes "proofs" that are correctly prepared and those with flaws in them, proofs that can be found in the mathematical tradition and those prepared by students. Examine the "proofs" presented here, judge their soundness, improve on them; in short, pretend you are the teacher and grade the material presented.

THEOREM 1. *The number* 1 *is the largest integer.*

Alleged Proof. Suppose the conclusion is false. Then let $n > 1$ be the largest integer. Multiplying both sides of this inequality by n, yields $n^2 > n$. This is a contradiction, because n^2 is another integer larger than n. Thus, the theorem is proved. ■

THEOREM 2. *The value of the expression* $\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\ldots}}}}}$ *is* 3.

Alleged Proof. Let

$$x = \sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\ldots}}}}}$$

Then

$$x^2 = 3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\ldots}}}}}$$

This implies that $x^2 = 3x$. As $x \neq 0$, the only solution is $x = 3$. ■

Theorem 3. *There is no prime number larger than* 12 *million.*

Alleged Proof. Let x be a number larger than 12 million.

Case 1. The number x is even. Then it is a multiple of 2, thus it is not prime.

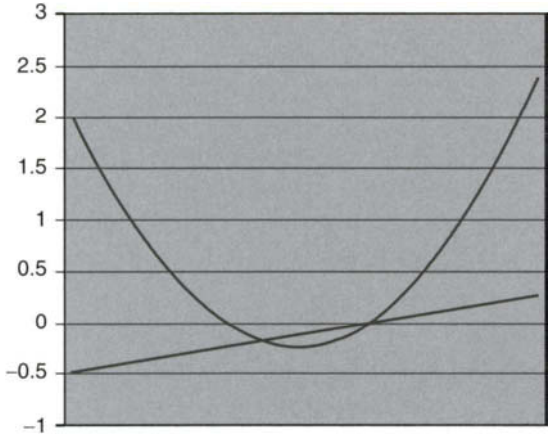
Case 2. The number x is odd. As x is odd, the two numbers $a = (x+1)/2$ and $b = (x-1)/2$ are both integers. In addition, $0 < b < a$. By performing some algebra, one can prove that:

$$x = (a+b)(a-b).$$

As x is the product of two integers (namely, $a+b$ and $a-b$), one can conclude that x is not a prime number. Thus, there is no prime number larger than 12 million. ■

Theorem 4. *The graphs of the curves represented by the equations* $y = x^2 - x$ *and* $y = \frac{1}{4}x - \frac{1}{4}$ *have at most two points in common.*

Alleged Proof 1. Graph both equations as shown here. The graph clearly shows that there are two points in common. ■



Alleged Proof 2. We need to solve the equation:

$$x^2 - x = \frac{1}{4}x - \frac{1}{4}.$$

If we write all the terms in the left-hand side, we get:

$$x^2 - \frac{5}{4}x + \frac{1}{4} = 0.$$

So,

$$x = \frac{5/4 \pm \sqrt{25/16 - 1}}{2} = \frac{5/4 \pm 3/4}{2}.$$

The two solutions are $x = 1$ and $x = 1/4$. ■

Alleged Proof 3. A second-degree equation has at most two solutions. So, the graphs meet in at most two points. ■

Alleged Proof 4. To show that the statement is true, one needs to show that the two graphs have in common no points, one point, or two points. To find the coordinates of the points one must solve the equation:

$$x^2 - x = \frac{1}{4}x - \frac{1}{4}.$$

By factoring, the equation can be rewritten as:

$$x(x - 1) = \frac{1}{4}(x - 1).$$

By dividing both sides, one gets:

$$x = \frac{1}{4}.$$

Therefore, the graphs of the two equations have in common only the point with coordinates (1/4, −3/16). Thus, the given statement is true. ■

Theorem 5. *Let n be an integer with $n \geq 1$. Then $n^2 - n$ is always even.*

Alleged Proof 1. We will use proof by induction.

Step 1: Is the statement true when $n = 1$?

$$1^2 - 1 = 0$$

Because 0 is an even number, the statement is true for the smallest number included in the statement.

Step 2: Assume that $k^2 - k$ is even for an integer k.
Step 3: Let's prove that the statement is true for $k+1$.

$$(k+1)^2 - (k+1) = k^2 + 2k + 1 - k - 1 = (k^2 - k) + 2k.$$

Because $k^2 - k$ is even and so is $2k$, we can conclude that the statement is true for $k+1$. ■

Alleged Proof 2. We will use proof by cases.

Case 1. Let n be an even number. Then $n = 2k$ for some integer number k. Thus,

$$\begin{aligned} n^2 - n &= (2k)^2 - 2k \\ &= 4k^2 - 2k \\ &= 2(2k^2 - k). \end{aligned}$$

Because the number $2k^2 - k$ is an integer, $n^2 - n$ is even.

Case 2. Let n be an odd number. Then $n = 2k + 1$ for some integer number k. Thus,

$$\begin{aligned} n^2 - n &= (2k+1)^2 - (2k+1) \\ &= 4k^2 + 4k + 1 - 2k - 1 \\ &= 2(2k^2 + k). \end{aligned}$$

Because the number $2k^2 + k$ is an integer, $n^2 - n$ is even. ■

Alleged Proof 3. We can write:

$$n^2 - n = n(n-1).$$

The product of two consecutive numbers is always even, so $n^2 - n$ is even. ■

Alleged Proof 4. Assume there is a positive integer s such that $s^2 - s$ is odd. Then we can write:

$$s^2 - s = 2k + 1$$

where k is an integer and $k \geq 0$. Then,

$$s = \frac{1 \pm \sqrt{8k+5}}{2}.$$

Because s is positive, we will only consider:

$$s = \frac{1 + \sqrt{8k+5}}{2}.$$

Clearly, $8k+5$ is never a perfect square (it is equal to 5, 13, 21, 37,...). Therefore, s is not an integer, and the given statement is true. ■

THEOREM 6. *All math books have the same number of pages.*

Alleged Proof. We will prove by induction on n that all sets of n math books have the same number of pages.

Step 1: Let $n = 1$. If X is a set of one math book, then all math books in X have the same number of pages.

Step 2: Assume that in every set of n math books all the books have the same number of pages.

Step 3: Now suppose that X is a set of $n+1$ math books. To show that all books in X have the same number of pages, it suffices to show that, if a and b are any two books in X, then a has the same number of pages as b. Let Y be the collection of all books in X, except for a. Let Z be the collection of all books in X, except for b. Then both Y and Z are collections of n books. By the inductive hypothesis, all books in Y have the same number of pages, and all books in Z have the same number of pages. So, if c is a book in both Y and Z, it will have the same number of pages as a and b. Therefore, a has the same number of pages as b. ■

THEOREM 7. *If $n \geq 0$ and a is a fixed nonzero real number, then $a^n = 1$.*

Alleged Proof. By mathematical induction.

Step 1: Let $n = 0$. Then, by definition, $a^0 = 1$.

Step 2: Assume that $a^k = 1$ for all $0 \leq k \leq n$ (strong inductive hypothesis).

Step 3: Let us work on $n+1$. By the rules of algebra:

$$a^{n+1} = \frac{a^n \times a^n}{a^{n-1}} = \frac{1 \times 1}{1} = 1.$$

Therefore, the conclusion is proved. ■

THEOREM 8. *The number $\sqrt{2}$ is irrational.*

Alleged Proof. Let us assume that $\sqrt{2}$ is a rational number. Then $\sqrt{2} = a/b$, with a and b positive integers. Thus,

$$\frac{a^2}{b^2} = 2.$$

Therefore,

$$a^2 = 2b^2.$$

This implies that a^2 is an even number; therefore, a is an even number. So, $a = 2a_1$, with a_1 integer positive number and $a_1 < a$.

This yields $4a_1^2 = 2b^2$; that is, $2a_1^2 = b^2$. Therefore, b^2 is an even number, which implies that b is an even number. So, $b = 2b_1$, with b_1 integer positive number and $b_1 < b$. Thus,

$$\sqrt{2} = \frac{a}{b} = \frac{2a_1}{2b_1} = \frac{a_1}{b_1}.$$

Because $\sqrt{2} = a_1/b_1$, we can repeat the process above and write $a_1 = 2a_2$ and $b_1 = 2b_2$ where a_2 and b_2 are positive integers, $b_2 < b_1$, and $a_2 < a_1$.

If this process is repeated k times, we can construct two sequences of integer positive numbers:

$$0 < a_k < a_{k-1} < \cdots < a_2 < a_1 < a$$
$$0 < b_k < b_{k-1} < \cdots < b_2 < b_1 < b.$$

If $k > b$, we have reached a contradiction. Therefore, $\sqrt{2}$ is an irrational number. ■

THEOREM 9. *Let A, B, and C be any three subsets of a universal set U. Then $A \cup (B \cap C') = (A \cup B) \cup (A \cap C')$.*

Alleged Proof. $x \in A \cup (B \cap C')$ if and only if either $x \in A$ or $x \in B \cap C'$ if and only if either $x \in A$ or $x \in B$ and $x \notin C$ if and only if either $x \in A$ or $x \in B$ or $x \in A$ and $x \notin C$ if and only if either $x \in A \cup B$ or $x \in A \cap C'$ if and only if $x \in (A \cup B) \cup (A \cap C')$. ■

Theorem 10. *The sum of the cubes of three consecutive integers is divisible by* 9.

Alleged Proof. For the base step observe that $0^3 + 1^3 + 2^3 = 9$ is indeed a multiple of 9. Suppose that $n^3 + (n+1)^3 + (n+2)^3 = 9k$ for some integer number k. Then:

$$\begin{aligned}(n+1)^3 + (n+2)^3 + (n+3)^3 &= n^3 + (n+1)^3 + (n+2)^3 + (n+3)^3 - n^3 \\ &= 9k + 9n^2 + 27n + 27 \\ &= 9(k + n^2 + 3n + 3),\end{aligned}$$

which is a multiple of 9. ■

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Solutions for the Exercises at the End of the Sections and the Review Exercises

WARNING: The following solutions for the exercises listed in this book should be used only as a guide. If you have not read the section "To the Reader" in the front of the book, here it is once more!

Learning to construct proofs is like learning to play tennis. It is useful to have someone teaching us the basics and it is useful to look at someone playing, but we need to get onto the court and play if we really want to learn. Therefore, we suggest that you, the reader, set aside a minimum time limit for yourself to construct a proof without looking at the solution (as a starting point, you could give yourself one hour and then adjust this limit to fit your ability). If you do not succeed, read only the first few lines of the proof presented here, and then try again to complete the proof on your own. If you are not able to do so, read a few more lines and try once more. If you need to read the entire proof, make sure that you understand it, and after a few days try the exercise again on your own.

The discussion part of a proof is enclosed in parentheses.

SOLUTIONS FOR THE EXERCISES AT THE END OF THE SECTIONS

Basic Techniques To Prove If/Then Statements

1. There exists at least one real number for which the function f is not defined. (Or: The function f is not defined for all real numbers.)
2. Let x and y be two numbers. There is no rational number z such that $x+z=y$. (Or: Let x and y be two numbers. Then $x+z\neq y$ for all rational numbers z.)
3. The function f does not have the property that for any two distinct real numbers x and $y, f(x)\neq f(y)$. (Or: There exist at least two distinct numbers x and y for which $f(x)=f(y)$.)
4. The equation $P(x) = 0$ has at least two solutions. (Or: The equation $P(x) = 0$ has more than one solution.)
5. There is at least one nonzero real number that does not have a nonzero opposite.
6. Either: (i) There exists a number $n > 0$ for which there is no number $M_n > 0$ such that $f(x) > n$ for all numbers x with $x > M_n$; or (ii) there exists a number $n > 0$ such that for every $M_n > 0$ there is at least one x with $x > M_n$ and $f(x) \leq n$.
7. There exists at least one number satisfying the equation $P(x) = Q(x)$ such that $|x| \geq 5$.
8. Compare this statement with statement 4. In this case, we do not know whether a solution exists at all. So the answer is: Either the equation $P(x) = 0$ has no solution or it has at least two solutions.
9. The function f is not continuous at the point c if there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an x with $|x-c| < \delta$ and $|f(x)-f(c)| \geq \varepsilon$.
10. There exists at least one real number x_0 such that $f(x_0)$ is an irrational number. (Or: The function $f(x)$ is not rational for every real number x.)
11. (a) If x is an integer not divisible by 2, then x is not divisible by 6. (b) If x is an integer divisible by 2, then x is divisible by 6. (c) If x is an integer not divisible by 6, then x is not divisible by 2.
12. (a) If the diagonals of a quadrilateral bisect, then the quadrilateral is a parallelogram. (b) If the diagonals of a quadrilateral do not bisect, then the quadrilateral is not a parallelogram. (c) If a quadrilateral is a parallelogram, then its diagonals bisect.

13. (a) If there is at least one i with $0 \leq i \leq n$ for which $a_i \neq b_i$, then there is at least one number x for which $P(x)$ and $Q(x)$ are not equal. (b) If $a_i = b_i$ for all i, with $0 \leq i \leq n$, then $P(x)$ and $Q(x)$ are equal for all real numbers. (c) If there exists at least one real number for which $P(x)$ and $Q(x)$ are not equal, then there is at least one i with $0 \leq i \leq n$ for which $a_i \neq b_i$.

14. (a) If the product of two integer numbers is not odd, then at least one of the integer numbers is not odd (or: the two integer numbers are not both odd). (b) If the product of two integer numbers is odd, the two integer numbers are odd. (c) If two integer numbers are not both odd, their product is not odd.

15. (a) If two numbers are both not even, then their product is not even. (b) If at least one of two integer numbers is even, their product is even. (c) If the product of two integer numbers is not even, then both (all) numbers are not even.

16. Let x and y be any two real numbers such that $x \leq y$. Can we prove that $f \circ g(x) \leq f \circ g(y)$? Because g is nondecreasing, $s = g(x) \leq g(y) = t$. Because f is nondecreasing $f(s) \leq f(t)$; that is, $f(g(x)) = f(g(y))$. So, the statement is true.

17. (Because we cannot directly check all products between rational and irrational numbers, we could consider using the contrapositive of the original statement.) Let's assume that xy is rational. Then $xy = a/b$ where a and b are integers and $b \neq 0$. Because x is a nonzero rational number, $x = c/d$, where with c and d integers, $d \neq 0$, and $c \neq 0$. What can we find out about y? Because of our assumption, $(c/d)y = a/b$. Because $c/d \neq 0$, we can multiply both sides of the equation by its inverse, d/c, and we obtain $y = (ad)/(bc)$. The numbers a, b, c, and d are integers, and $bc \neq 0$ because $b \neq 0$ and $c \neq 0$. Therefore, y is a rational number. Thus, we proved that the contrapositive of our original statement is true, so the original statement is true as well.

18. For the sake of simplicity, let us assume that n is positive. Because n has at least three digits, we can write $n = rst \ldots cba$, where r, s, $t, \ldots$, b, a represent the digits; therefore, they are all numbers between 0 and 9, and $r \neq 0$. Because we have some information about the number formed by the two rightmost digits, we will isolate them and write $n = (rst \ldots c) \times 100 + ba$. By hypothesis, ba is divisible by 4, so $ba = 4t$ for some integer number t. Thus, $n = (rst \ldots c) \times 100 + 4t = 4[25(rst \ldots c) + t]$. The number $25(rst \ldots c) + t$ is an integer. This proves that n is divisible by 4. Repeat the proof for the case in which n is a negative number.

19. By hypothesis $(a+b)^2 = a^2 + b^2$. But, by the rules of algebra $(a+b)^2 = a^2 + 2ab + b^2$. Thus, we obtain that, for all real numbers b, $a^2 + 2ab + b^2 = a^2 + b^2$. This implies that $2ab = 0$ for all real numbers b. In particular, this equality is true when $b \neq 0$. Thus, we can divide the equality $2ab = 0$ by $2b$ and obtain $a = 0$.

20. (Because it is impossible to check directly all prime numbers that can be written in the form $2^n - 1$ to see if the corresponding exponent n is indeed a prime number, we will try to use the contrapositive of the original statement.) We will assume that n is not a prime number. Then n is divisible by at least another number t different from n and 1. So, $t \geq 2$. Thus, $n = tq$ where q is some positive integer, $q \neq 0$ (because $n \neq 0$), $q \neq 1$ (because $t \neq n$), and $q \neq n$ (because $t \neq 1$). Therefore, we can write:

$$2^n - 1 = 2^{tq} - 1 = (2^q)^t - 1.$$

We can now use factorization techniques to obtain:

$$2^n - 1 = (2^q)^t - 1 = (2^q - 1)\left[(2^q)^{t-1} + (2^q)^{t-2} + \cdots + 1\right].$$

This equality shows that $2^n - 1$ is not a prime number because it can be written as the product of two numbers, and neither one of these numbers is 1. (Why? Look at all the information listed above regarding t and q.) Because we proved that the contrapositive of the original statement is true, we can conclude that the original statement is true and n must be a prime number.

21. Let n be a four-digit palindrome number. To prove that it is divisible by 11, we need to show that $n = 11t$ for some positive integer t. Because n is a four-digit palindrome number, we can write $n = xyyx$, where x and y are integer numbers between 0 and 9, and $x \neq 0$. We can now try to separate the digits of n. Thus,

$$\begin{aligned} n = xyyx &= 1000x + 100y + 10y + x \\ &= 1001x + 110y = 11(91x + 10y). \end{aligned}$$

Because $t = 91x + 10y$ is a positive integer, we proved that n is divisible by 11.

22. We know that $x \neq c$. So, the denominator of the fraction is a nonzero number, and the fraction is well defined. We want to prove that the number

$$\frac{f(c) - f(x)}{c - x}$$

is nonnegative. (Because of the algebraic properties that determine the sign of a fraction, we need to prove that both the numerator and the denominator have the same sign, and consider the possibility that the numerator equals 0.) Because $c - x \neq 0$, there are two possible cases:

i. $c - x > 0$

ii. $c - x < 0$.

In the first case (namely, $c - x > 0$), $c > x$. Because the function f is nondecreasing, $f(c) \geq f(x)$. Thus, $f(c) - f(x) \geq 0$ and $c - x > 0$. This implies that the fraction $((f(c) - f(x))/(c - x))$ is nonnegative. In the second case (namely, $c - x < 0$), $c < x$. Because the function f is nondecreasing, $f(c) \leq f(x)$. Thus, $f(c) - f(x) \leq 0$ and $c - x < 0$. This implies that the fraction $((f(c) - f(x))/(c - x))$ is nonnegative.

23. To prove that f is one-to-one, we have to prove that, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

i. *Direct method.* Because $x_1 \neq x_2$ and $m \neq 0$, it follows that $mx_1 \neq mx_2$. Thus, $mx_1 + b \neq mx_2 + b$. So $f(x_1) \neq f(x_2)$.

ii. *Contrapositive.* We will assume that $f(x_1) = f(x_2)$. This assumption allows us to set up an equality and use it as a starting point. Because $f(x_1) = f(x_2)$, it follows that $mx_1 + b = mx_2 + b$. This equality implies that $mx_1 = mx_2$. Because $m \neq 0$, we can divide both sides of the equality by m, and we obtain $x_1 = x_2$. Thus, we proved that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. This is equivalent to proving that, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. So the function f is one-to-one.

24. We have to prove that for every real number y there is at least one real number x such that $f(g(x)) = y$. By hypothesis f is onto. Therefore, there exists at least one real number z such that $f(z) = y$. The function g is onto. Therefore, there exists at least one real number t such that $g(t) = z$. Therefore, $f(g(t)) = f(z) = y$, and this proves that the function $f \circ g$ is onto.

"If and Only If" or Equivalence Theorems

1. *Part 1.* Assume that f is a nonincreasing function. We want to show that $((f(x) - f(c))/(x - c) \leq 0)$ for all c and x in the domain of f with $c \neq x$. Because $c \neq x$, there are two possibilities: either $x < c$ or $x > c$. If $x < c$, because f is nonincreasing, it follows that $f(x) \geq f(c)$. These two inequalities can be rewritten as:

$$\text{If } x - c < 0, \text{ then } f(x) - f(c) \geq 0; \text{ therefore, } \frac{f(x) - f(c)}{x - c} \leq 0.$$

If $x > c$, because f is nonincreasing, it follows that $f(x) \leq f(c)$.

These two inequalities can be rewritten as:

$$\text{If } x - c > 0, \text{ then } f(x) - f(c) \leq 0; \text{ therefore, } \frac{f(x) - f(c)}{x - c} \leq 0.$$

Part 2. The hypothesis is that $((f(x) - f(c))/(x - c) \leq 0$ for all c and x in the domain of f with $c \neq x$. Does this inequality imply that f is a nonincreasing function? A quotient between two real number is nonpositive when one of the two numbers is negative and the other is positive (or the dividend is zero). Suppose that the denominator is positive. Then the numerator must be either negative or zero. This means that if $x - c > 0$, then $f(x) - f(c) \leq 0$. Therefore, if $x > c$, it follows that $f(x) \leq f(c)$. Suppose that the denominator is negative. Then the numerator is either positive or zero. This means that if $x - c < 0$, then $f(x) - f(c) \geq 0$. Therefore, if $x < c$, it follows that $f(x) \geq f(c)$.

So, in either case we can conclude that f is a nonincreasing function.

2. *Part 1.* Let a and b be two odd numbers. Then $a = 2k + 1$ and $b = 2t + 1$, with k and t integers, and $ab = (2k + 1)(2t + 1) = 4kt + 2k + 2t + 1 = 2(2k + k + t) + 1$. Because the number $2kt + k + t$ is an integer, then ab is an odd number.

   Part 2. We want to prove that, if ab is an odd number, then a and b are both odd. We will use the contrapositive of this statement. Assume that either a or b is an even number. Will this imply that the product ab is an even number? Without loss of generality, let us assume that a is even. Then $a = 2k$, where k is an integer. Thus, $ab = 2(kb)$, where kb is an integer. So ab is even. Because the contrapositive of the original statement is true, the statement itself is true as well.

3. *Part 1.* Assume that n is divisible by 3. Thus, $n = 3k$, with k integer. Therefore, $n^2 = 9k^2 = 3(3k^2)$. Because $3k^2$ is an integer, we conclude that n^2 is a multiple of 3.

   Part 2. We will prove this part using the contrapositive. Assume that n is not divisible by 3. Then $n = 3t + m$ when $m = 1$ or $m = 2$. Therefore, $n^2 = (3t + m)^2 = 9t^2 + 6tm + m^2 = 3(3t^2 + 2tm) + m^2$. The number $3t^2 + 2tm$ is an integer. The number m^2 is equal to either 1 or to 4, and it is not divisible by 3. Thus, n^2 is not a multiple of 3. Because the contrapositive of the original statement is true, the statement itself is true as well.

4. We will prove that (i) implies (ii), (ii) is equivalent to (iii), and (iii) implies (i). (This is only one of several possible ways of constructing this proof.)

   (i) implies (ii): We have to prove that $a^s < a^r$ for all real numbers $a > 1$, or, equivalently, that $a^s - a^r < 0$ for all real numbers $a > 1$.

Because $r > s$, we can write $a^s - a^r = a^s(1 - a^{r-s}) = a^s(1 - a^t)$, where $t = r - s$ is a positive number. Let us consider the two factors of this product: a^s and $1 - a^t$. The first factor, a^s, is positive because $a > 1$. The second factor, $1 - a^t$, is negative for the same reason. Therefore, $a^s(1 - a^t) < 0$, which is the conclusion we wanted to obtain.

(ii) implies (iii): Let a be a real number such that $a < 1$. The hypothesis we are using is about numbers larger than 1. So, we need to relate a to a number larger than 1. The inverse of any number smaller than 1 is a number larger than 1. If we consider $b = a^{-1}$, then $b > 1$. Moreover $a = (a^{-1})^{-1} = b^{-1}$ when $b > 1$. Then, by hypothesis, $b^s < b^r$. Therefore, $(a^{-1})^s < (a^{-1})^r$; that is, $\frac{1}{a^s} < \frac{1}{a^r}$. This implies that $a^s > a^r$.

(iii) implies (ii): Any number larger than 1 is the inverse of a number smaller than 1. Thus, $a = (a^{-1})^{-1} = b^{-1}$ when $b < 1$. Then, by hypothesis, $b^r < b^s$. Therefore, $(a^{-1})^r < (a^{-1})^s$; that is, $\frac{1}{a^r} < \frac{1}{a^s}$. This implies that $a^s < a^r$.

(iii) implies (i): By hypothesis $a^r < a^s$ for all real numbers $a < 1$; or $a^r - a^s < 0$ for all real numbers $a < 1$. We can rewrite this difference as $a^r - a^s = a^r(1 - a^{s-r})$. This product is negative and its first factor is positive. Therefore, the other factor, $(1 - a^{s-r})$, must be negative. So $1 < a^{s-r}$. Because $a < 1$, this will happen only if $s - r < 0$. Thus, $s < r$.

5. *(i) implies (ii):* Already proved; see Example 3 in this section.

   (ii) implies (iii): From the inequality $(a + b)/2 > a$, we obtain $a + b > 2a$. This implies $b > a$. Because $b > a$, it follows that $2b > a + b$. Thus, $b > (a + b)/2$.

   (iii) implies (i): Already proved; see Example 3 in this section.

6. *(i) implies (ii):* The numbers x and y are both negative. Thus, by definition of absolute value, $|x| = -x$ and $|y| = -y$. The inequality $x < y$ implies $-x > -y$. So $|x| > |y|$.

   (ii) implies (i): Because x and y are both negative numbers, $|x| = -x$ and $|y| = -y$. The inequality $|x| > |y|$ implies $-x > -y$. Therefore, $x < y$.

   (iii) implies (i): By hypothesis $x^2 > y^2$. So $x^2 - y^2 > 0$, or $(x - y)(x + y) > 0$. The number $x + y$ is negative because x and y are both negative numbers. The product $(x - y)(x + y)$ can be positive only if the number $x - y$ is negative as well. Then, $x - y < 0$, or $x < y$.

   (i) implies (iii): Because $x < y$, it follows that $x - y < 0$. Because we want to obtain information about $x^2 - y^2$, we can use factorization techniques to write $x^2 - y^2 = (x - y)(x + y)$. The first factor is negative by hypothesis. The second factor is negative because it is the sum of two negative numbers. Therefore, $x^2 - y^2 > 0$, or $x^2 > y^2$.

7. Assume that (x_0, y_0) is a solution of S_1. Is it a solution of S_2? By definition of solution, (x_0, y_0) satisfies both equations of the system S_1. So (x_0, y_0) satisfies the first equation of S_2. Thus, we only need to prove that it satisfies the second equation of S_2. By rearranging the terms we can write $(a_1 + ba_2)x_0 + (b_1 + bb_2)y_0 = (a_1x_0 + b_1y_0) + b(a_2x_0 + b_2y_0)$. Because (x_0, y_0) is a solution of S_1, $a_1x_0 + b_1y_0 = c_1$, $a_2x_0 + b_2y_0 = c_2$. Thus, $(a_1 + ba_2)x_0 + (b_1 + bb_2)y_0 = c_1 + bc_2$. So (x_0, y_0) is a solution of S_2. Assume now that (x_0, y_0) is a solution of S_2. Is it a solution of S_1? By definition of solution, (x_0, y_0) satisfies both equations of system S_2. Thus, (x_0, y_0) satisfies the first equation of S_1 as well. Therefore, we must prove that it satisfies the second equation of S_1. By hypothesis, $(a_1 + ba_2)x_0 + (b_1 + bb_2)y_0 = c_1 + bc_2$. We can rewrite the left-hand side of the second equation of S_2 as $(a_1 + ba_2)x_0 + (b_1 + bb_2)y_0 = (a_1x_0 + b_1y_0) + b(a_2x_0 + b_2y_0)$. Because (x_0, y_0) is a solution of S_2, the left-hand side of the equation is equal to $c_1 + bc_2$, and the first expression in the parentheses on the right-hand side of the equation is equal to c_1. Therefore, we obtain $c_1 + bc_2 = c_1 + b(a_2x_0 + b_2y_0)$. This equality implies that $a_2x_0 + b_2y_0 = c_2$, as b is a nonzero number. This proves that (x_0, y_0) is a solution of S_1.

Use of Counterexamples

1. (Let us consider the statement. It seems to suggest that the "growth" of f should be cancelled by the "drop" of g. But the two functions could increase and decrease at different rates. Therefore, the statement does not seem to be true.) Let us construct a counterexample, using simple functions. We could try to use linear functions. Consider $f(x) = x + 1$ and $g(x) = -3x$. Clearly, f is increasing and g is decreasing. (If you wish to do so, prove these claims.) Their sum is $h(x) = -2x + 1$, which is decreasing.

2. There might be an angle in the first quadrant for which $2 \sin t = \sin 2t$. But the equality is not true for all the angles in the first quadrant. Consider $t = \pi/4$. The left-hand side of the equation equals $2\sin(\pi/4) = 2(1/\sqrt{2}) = \sqrt{2}$. The right-hand side equals $\sin(\pi/2) = 1$. Thus, the statement is false.

3. (It might seem plausible that $y = P(x)$ is always negative, because its leading coefficient is negative [it is -1]. But, when the values of the variable x are not too large, the value of the monomial $-x^2$ can be smaller than the value of the monomial $2x$. This remark seems to suggest that for positive values of x that are not too large the variable $y = P(x)$ might be positive [or at least nonnegative].) We will look for a value of the variable x that makes the polynomial

nonnegative: $P(1) = -(1)^2 + 2(1) - (3/4) = 1/4$. Another way to prove that the statement is false is to construct the graph of the polynomial $P(x)$, and to observe that the graph is not completely located below the x-axis.

4. The statement seems to be true. But, if $x = 1$, then $y = 1$. Thus, we have found a counterexample. The statement becomes true if we either change the hypothesis to "the reciprocal of a number $x > 1$" or the conclusion to "$0 < y \leq 1$."

5. If $n = 1$, then $3^1 + 2 = 5$, which is a prime number.
If $n = 2$, then $3^2 + 2 = 11$, which is a prime number.
If $n = 3$, then $3^3 + 2 = 29$, which is a prime number.
If $n = 4$, then $3^4 + 2 = 83$, which is a prime number.
If $n = 5$, then $3^5 + 2 = 245$, which is not a prime number. Therefore, the statement is false.

6. (The functions $f \circ g$ and $f \circ h$ are equal if and only if $f \circ g(x) = f \circ h(x)$ for all the values of the variable x. By definition of composition of function, this equality can be rewritten as $f(g(x)) = f(h(x))$. From this, can we conclude that $g(x) = h(x)$? Or could we find a function f such that $f(g(x)) = f(h(x))$ even if $g(x) \neq h(x)$? The answer does not seem to be obvious. Let's look for some functions that might provide a counterexample. Keep in mind that counterexamples do not need to be "complicated.") Let's try to use $g(x) = x$ and $h(x) = -x$. Is it possible to choose a function f such that $f(g(x)) = f(h(x))$? Given our choices of the functions g and h, this equality becomes $f(x) = f(-x)$. So we need a function that assigns the same output to a number and its opposite. What about $f(x) = x^2$? We will now check to see if we really have found a counterexample:

$$f \circ g(x) = f(g(x)) = f(x) = x^2$$
$$f \circ h(x) = f(h(x)) = f(-x) = (-x)^2 = x^2.$$

So the equality $f(g(x)) = f(h(x))$ holds, but the functions g and h are not equal. Using the same choices for g and h, we can use $f(x) = \cos x$, or $f(x) = x^4$, or $f(x) = x^6$, or any other even function.

7. Let us try to prove this statement. Let n be the smallest of the five consecutive integers we are going to add. So the other four numbers can be written as $n + 1$, $n + 2$, $n + 3$, and $n + 4$. The sum of these five numbers is $S = n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10$. This number is divisible by 5, so the statement is true.

8. We will try to construct a proof of the statement. The inequality $f(x) \leq g(x)$ is equivalent to the inequality $f(x) - g(x) \leq 0$. Therefore, we can concentrate on proving that $f(x) - g(x) \leq 0$ for all real numbers $x \geq 0$. Using the formulas for the functions, we obtain:

$$f(x) - g(x) = x^2 - x^4 = x^2(1 - x^2) = x^2(1 - x)(1 + x).$$

Is this product smaller than or equal to zero for all real numbers $x \geq 0$? The product is equal to zero for $x = 0$ and $x = 1$. (We are not considering $x = -1$ because we are using only nonnegative numbers.) What happens to the product $x^2(1 - x)(1 + x)$ if x is neither 0 nor 1? The number x^2 is always positive. Because $x \geq 0$, $1 + x \geq 1$. So this factor is always positive. Then the sign of the product is determined by the factor $1 - x$. This factor is less than or equal to zero when $x \geq 1$. Therefore, $x^2(1 - x)(1 + x) \leq 0$ only when $x \geq 1$. So $f(x) - g(x) \leq 0$ only if $x \geq 1$, and not for all $x \geq 0$. Thus, the statement is false. Can we find a counterexample? Consider $x = 0.2$. Then $x^2 = 0.04$ and $x^4 = 0.0016$. Therefore, in this case $x^2 > x^4$, and the statement is false.

9. Let n be the smallest of the four counting numbers we are considering. Then the other three numbers are $n + 1$, $n + 2$, and $n + 3$. When we add these numbers we obtain:

$$S = n + (n + 1) + (n + 2) + (n + 3) = 4n + 6.$$

The number S is not *always* divisible by 4. Indeed, if $n = 1$, $S = 10$. So, we have found a counterexample. The sum of the four consecutive integers 1, 2, 3, and 4 is not divisible by 4. The given statement is false. *Note:* S is *always* divisible by 2.

10. (This statement seems to be similar to the statement: "The sum of two odd numbers is an even number." But similarity is never a proof, and statements that sound similar can have very different meanings. So, we must try to construct either a proof or a counterexample.) To prove that the function $f + g$ is even we need to prove that:

$$(f + g)(x) = (f + g)(-x)$$

for all real numbers. By definition of $f + g$:

$$(f + g)(x) = f(x) + g(x)$$
$$(f + g)(-x) = f(-x) + g(-x).$$

Because f and g are odd functions, $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Thus, we obtain:

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x) = (-f(x)) + (-g(x)) \\ &= -[f(x) + g(x)] = -(f+g)(x).\end{aligned}$$

Therefore, $(f+g)(-x) = -(f+g)(x)$, and $-(f+g)(x)$ in general is not equal to $(f+g)(x)$. So, the equality $(f+g)(x) = (f+g)(-x)$ does not seem to be true for all real numbers. Can we find a counterexample? The functions $f(x) = x$ and $g(x) = 2x$ are two odd functions (check this claim). Their sum is the function $(f+g)(x) = 3x$, which is not even. (Moreover, $(f+g)(x) = 3x$ is odd.)

11. The function f/g is not going to be defined for all values of x, in general. Indeed, it is not defined for all values of x that are zeros for the function g. Thus, in general the statement is false. The fact that f/g is either even or odd is not relevant. As an explicit counterexample, consider $f(x) = x$ and $g(x) = x^3 - x^5$. Try to prove the following statement: Let f and g be two odd functions defined for all real numbers. Their quotient, the function f/g defined as $((f/g)(x)) = (f(x)/g(x))$, is an even function defined for all real numbers for which $g(x) \neq 0$.

12. Let $n = xyzzyx$, with $1 \leq x \leq 9$, $0 \leq y \leq 9$, and $0 \leq z \leq 9$. Then we can write:

$$n = x + 10y + 100z + 1{,}000z + 10{,}000y + 100{,}000x.$$

Therefore, $n = 1{,}100z + 10{,}010y + 100{,}001x = 11 \times 100z + 11 \times 910y + 11 \times 9091x$. Thus, $n = 11(100z + 910y + 9091x)$. Because the number $100z + 910y + 9091x$ is an integer, we conclude that n is divisible by 11.

13. It is true that, if two numbers are rational, then their sum is rational; however, the converse of this statement is not true. If x is an irrational number, its opposite, $-x$, is irrational as well. Their sum is 0, which is a rational number. Thus, the sum of two numbers can be rational without either one of them being rational.

14. To prove that g is even, we need to show that $g(x) = g(-x)$ for all real numbers x. Because f is an odd function, $f(-x) = -f(x)$, and $g(x) = (f(x))^2 = (-f(x))^2 = (f(-x))^2 = g(-x)$. Thus, g is even.

15. This statement is false. Consider $f(x) = x^2 + 1$. Then $g(x) = (x^2 + 1)^3$. If we use $x_1 = -1$ and $x_2 = 0$, then $x_1 < x_2$ but $g(x_1) > g(x_2)$.
Calculus approach: If we want to find the derivative of g, using the chain rule, we obtain $g'(x) = 3(f(x))^2 f'(x)$. The factor $3(f(x))^2$ is

always positive, but $f'(x)$ might be negative, even when $f(x)$ is positive. Consider the function $f(x) = x^2 + 1$. Its derivative is $f'(x) = 2x$, which is negative for $x < 0$.

Mathematical Induction

1. (Note that the sum on the left-hand side of the equation involves exactly k numbers.)

   a. Is the statement true for $k = 1$? Yes, because $1 = 2^1 - 1$.
   b. Let us assume that the equality is true for $k = n$. So

$$\underbrace{1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}}_{n \text{ numbers}} = 2^n - 1.$$

   c. Let us check if the equality holds for $n + 1$:

$$\underbrace{1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} + 2^n}_{(n+1) \text{ numbers}} = 2^{n+1} - 1.$$

   Using the associative property of addition we can write:

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} + 2^n = (1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}) + 2^n.$$

   If we now use the inductive hypothesis from part b, we obtain

$$\begin{aligned} 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} + 2^n &= (2^n - 1) + 2^n \\ &= 2 \times 2^n - 1 = 2^{n+1} - 1. \end{aligned}$$

   So the statement is true for $n + 1$. Thus, by the principle of mathematical induction, the statement is true for all $k \geq 1$.

2. Let us prove this statement by induction.

   a. We will begin by proving that the statement is true for $k = 1$. Indeed, when $k = 1$, $9^1 - 1 = 8$, and 8 is divisible by 8.
   b. Assume that the statement is true for $k = n$. So, $9^n - 1 = 8q$ for some integer number q.
   c. Prove that the statement is true for $n + 1$. By performing some algebra, we obtain

$$\begin{aligned} 9^{n+1} - 1 = 9^{n+1} - 1 &= 9(9^n) - 1 \\ &= 9(9^n - 1) + 9 - 1. \end{aligned}$$

When we use the inductive hypothesis, the equality becomes:

$$\begin{aligned} 9^{n+1} - 1 &= 9(9^n - 1) + 8 \\ &= 9(8q) + 8 = 8(9q + 1). \end{aligned}$$

Because $9q + 1$ is an integer number, it follows that $9^{n+1} - 1$ is divisible by 8. Thus, by the principle of mathematical induction, the statement is true for all $k \geq 1$.

There is another way to construct a proof without using mathematical induction. The basic tool is the factorization formula for the difference of two powers:

$$\begin{aligned} 9^k - 1 &= 9^k - 1^k \\ &= (9 - 1)(9^{k-1} + 9^{k-2} + \cdots + 1) \\ &= 8(9^{k-1} + 9^{k-2} + \cdots + 1) = 8s \end{aligned}$$

where s is an integer number. The formula used above requires that $k > 1$. So, we need to use a separate proof for $k = 1$. When $k = 1$, we have $9^1 - 1 = 8$, which is divisible by 8.

3. (i) Let us check whether the statement is true for $k = 1$. When $k = 1$, $2k = 2$. Therefore, we have only one number in the left-hand side of the equation. We obtain $2 = 1^2 + 1$, which is a true statement.
   (ii) Assume that the statement is true for $k = n$; that is, $2 + 4 + 6 + \cdots + 2n = n^2 + n$.
   (iii) Prove that the equality is true for $k = n + 1$. The last number in the left-hand side is $2(n + 1) = 2n + 2$. So, we need to add all the even numbers between 2 and $2n + 2$. The largest even number smaller than $2n + 2$ is $(2n + 2) - 2 = 2n$, because the difference between two consecutive even numbers is 2. Thus, we need to prove that $2 + 4 + 6 + \cdots + 2n + (2n + 2) = (n + 1)^2 + (n + 1)$. Using the associative property of addition and the inductive hypothesis, we obtain:

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2n + (2n + 2) \\ &= [2 + 4 + 6 + \cdots + 2n] + (2n + 2) \\ &= [n^2 + n] + (2n + 2) \\ &= (n^2 + 2n + 1) + (n + 1) \\ &= (n + 1)^2 + (n + 1). \end{aligned}$$

Thus, by the principle of mathematical induction, the given equality holds true for all $k \geq 1$.

4. a. We have to check whether the statement holds true for $k=3$:

$$(1+a)^3 = 1+3a+3a^2+a^3 > 1+3a^2.$$

The inequality is true because all the numbers used are positive.

b. Let us assume that the inequality is true for $k=n$; that is, assume that $(1+a)^n > 1+na^2$.

c. Let us check whether $(1+a)^{n+1} > 1+(n+1)a^2$. We can use rules of algebra and the inductive hypothesis to obtain

$$\begin{aligned}(1+a)^{n+1} &= (1+a)^n(1+a)\\ &> (1+na^2)(1+a)\\ &= 1+na^2+a+na^3\\ &> 1+a^2+na^3.\end{aligned}$$

Because $a > 1$, $a^3 > a^2$. So

$$\begin{aligned}(1+a)^{n+1} &> 1+na^2+na^3\\ &> 1+na^2+na^2\\ &> 1+na^2+a^2\\ &= 1+(n+1)a^2.\end{aligned}$$

Thus, by the principle of mathematical induction, the original statement is true.

5. a. We will check the equality for $k=1$. In this case, the left-hand side of the equation has only one term: 1/2. The right-hand side is equal to:

$$\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} - 1 = \frac{1}{2}.$$

So the equality is true for $k=1$.

b. Let us assume the equality holds true for $k=n$.

c. We will try to prove that

$$\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^{n+1} = \frac{1-(1/2)^{n+2}}{1-(1/2)} - 1.$$

Using the associative property of addition and the inductive hypothesis, we obtain

$$\begin{aligned}\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^{n+1} &= \left[\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^n\right]+\left(\frac{1}{2}\right)^{n+1}\\ &= \frac{1-(1/2)^{n+1}}{1-(1/2)}-1+\left(\frac{1}{2}\right)^{n+1}\\ &= \frac{1-(1/2)^{n+1}+(1/2)^{n+1}-(1/2)^{n+2}}{1-(1/2)}-1\\ &= \frac{1-(1/2)^{n+2}}{1-(1/2)}-1.\end{aligned}$$

So, by the principle of mathematical induction, the statement is true for all $k \geq 1$.

6. a. Check the inequality for $k=3$. In this case $3^2 = 9$ and $5(3!)=5(6)=30$. So the statement is true.
   b. Let us assume that the inequality holds true for an arbitrary $k=n$; that is, assume that $5n! \geq n^2$.
   c. Is $(n+1)^2 \leq 5(n+1)!$? By the properties of factorials, $(n+1)!=(n+1)n!$. So

$$\begin{aligned}5(n+1)! &= 5(n+1)n! = 5nn! + 5n!\\ &\geq 5nn! + n^2.\end{aligned}$$

   The fact that $n \geq 3$ implies that $n! \geq 3$. So $5nn! \geq 15n$. Thus,

$$\begin{aligned}5(n+1)! &\geq 5nn! + n^2\\ &\geq n^2 + 15n \geq n^2 + 3n\\ &= n^2 + 2n + n \geq n^2 + 2n + 1\\ &= (n+1)^2.\end{aligned}$$

   So, by the principle of mathematical induction, the inequality is true for all $n \geq 3$.

7. a. The statement is true for $n=1$ because $1^2 - 1 = 0$, and 0 is divisible by 4.
   b. Assume the statement is true for all the odd numbers from 1 to n, where n is odd. In particular, assume that $n^2 - 1 = 4m$ for some positive integer m.
   c. Is the statement true for the next odd number (namely, $n+2$)? Observe that $(n+2)^2 - 1=(n^2 - 1) + 4(n+1)$. Because $n^2 - 1=4m$ for some positive integer m, by the inductive hypothesis we have $(n+2)^2 - 1 = 4(m+n+1)$. Because the number $m+n+1$ is an

integer, $(n+2)^2 - 1$ is divisible by 4. Therefore, the statement is true for all $k \geq 1$ by the principle of mathematical induction.

There is another proof of this statement that does not use the principle of mathematical induction. Let n be an odd counting number. Then $n = 2k + 1$, where k is a counting number. Therefore,

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 = 4(k^2 + k). \end{aligned}$$

Because $k^2 + k$ is a counting number, $n^2 - 1$ is divisible by 4. One can observe that $n^2 - 1$ is divisible by 8 because $k^2 + k = k(k+1)$. The sum of two consecutive numbers is always an even number. Thus, $k^2 + k = k(k+1) = 2s$, and $n^2 - 1 = 8s$.

8. a. The statement is true for $k = 1$, as $10 - 1 = 9$.
   b. Assume it is true for a generic number n. Thus, $10^n - 1 = 9s$, where s is an integer.
   c. Is the statement true for the next number (namely, $n+1$)? Using algebra and the inductive hypothesis, we obtain:

$$\begin{aligned} 10^{n+1} - 1 &= (9+1)10^n - 1 \\ &= 9 \times 10^n + (10^n - 1) \\ &= 9 \times 10^n + 9s = 9(10^n + s). \end{aligned}$$

Therefore, the conclusion is true for all $k \geq 1$ by the principle of mathematical induction. It is possible to prove this statement using factorization techniques for differences of powers because $10^k - 1 = 10^k - 1^k$.

9. a. Check the statement for $n = 5$. Because $3^5 = 243$, the next to the last digit from the right is 4, an even number.
   b. Assume the statement is true for a generic number n. Thus, $3^n = a_k a_{k-1} \ldots a_1 a_0$, where $a_1 = 0, 2, 4, 6, 8$ and $a_0 = 1, 3, 7, 9$ (check the information regarding a_0).
   c. Is the statement true for the next number (namely, $n+1$)? We have:

$$3^{n+1} = 3(a_k a_{k-1} \ldots a_1 a_0) = b_t b_{t-1} \ldots b_1 b_0.$$

If $a_0 = 1$ or 3, then b_1 is the unit digit of $3a_1$. Because $3a_1$ is an even number (since a_1 is even), b_1 is even.

If $a_0 = 9$ or 7, then b_1 is the unit digit of $3a_1 + 2$, which is an even number, as a_1 is even.

Thus, the statement is true for all $n \geq 5$ by the principle of mathematical induction.

10. a. Check the statement for $n=2$:

$$A^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

b. Assume it is true for a generic number $n > 2$; that is, $A^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$.

c. Is the statement true for $n+1$? Using the associative property of multiplication of matrices, we can write $A^{n+1} = A \times A^n$. Thus, by the inductive hypothesis we obtain:

$$A^{n+1} = A \times A^n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n+1 & 1 \end{pmatrix}.$$

Therefore, the statement is true for all $n \geq 2$ by the principle of mathematical induction.

Existence Theorems

1. (We are trying to find a function defined for all real numbers. Usually polynomials are good candidates. But, in general, the range of a polynomial is much larger than the interval [0,1]. We could try to construct a rational function, because it is possible to use the denominator to control the growth of the function. But often rational functions are not defined for all real numbers. We could try using transcendental functions.) The functions $\sin x$ and $\cos x$ are bounded, because $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$, but their ranges include negative values as well. We could try to square them, or to consider their absolute values to obtain bounded and nonnegative functions. The functions $\sin^2 x$, $\cos^2 x$, $|\sin x|$, and $|\cos x|$ are functions defined for all real numbers and whose ranges are in the interval [0,1]. You can graph them to check this claim, if you wish. *Note:* The functions with the second power are differentiable; the ones with the absolute value are not differentiable.

2. Just use $n=2$. Then $2^2 + 7^2 = 53$, which is a prime number.

3. We are searching for a number b such that $ab = n$, where n is an integer. Because 0 is a rational number, we know that $a \neq 0$ and $b \neq 0$. Then we can solve the equation and use $b = na^{-1}$. So, for example, $b = a^{-1}$ satisfies the requirement. Because a is irrational, $a^{-1}, 2a^{-1}, 3a^{-1}, \ldots$ are irrational as well.

4. A second-degree polynomial $P(x)$ can be written as $P(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. We are looking for a second-degree polynomial satisfying the requirements $P(0) = -1$

and $P(-1)=2$. From the first condition, we obtain $P(0) = a(0)^2 + b(0) + c = c$. Thus, $c=-1$. From the second condition we obtain $P(-1) = a(-1)^2 + b(-1) + c = a - b + c$. Using the fact that $c=-1$, the second equation yields $a-b=3$. We have one equation and two variables. Thus, one variable will be used as a parameter. As an example, we can write $a=b+3$ and then choose any value we like for b. If $b=0$, we obtain the polynomial $P(x) = 3x^2 - 1$. Clearly, this is only one of infinitely many possibilities.

5. Because b^a will be a negative number, b must be negative. Indeed, powers of positive numbers are always positive. Moreover, a cannot be an even number, because an even exponent generates a positive result. The numbers a and b can be either fractions or integer numbers. Let us try $b=-27$ and $a=1/3$. Then $a^b = (1/3)^{-27} = 3^{27}$, which is a positive integer number, and $b^a = (-27)^{1/3} = -3$, which is a negative integer.

6. We can consider two cases: either $a_n>0$ or $a_n<0$. Let us assume that $a_n>0$. When the variable x has a very large positive value, the value of $P(x)$ will be positive, because the leading term, $a_n x^n$, will overpower all the other terms (*i.e.*, $\lim_{x\to+\infty} P(x) = +\infty$). When the variable x is negative, and very large in absolute value, the value of $P(x)$ will be negative for the same reason (*i.e.*, $\lim_{x\to-\infty} P(x) = -\infty$). Polynomials are continuous functions. Therefore, by the Intermediate Value Theorem, there exists a value of x for which $P(x) = 0$. We can prove in a similar way that the statement is true when $a_n<0$.

7. This can be either a theoretical or constructive proof. Let us use the constructive approach. We can write $a = p/q$ and $b = n/m$, where m, n, p, and q are integer numbers, $q\neq 0$, and $m\neq 0$. Let $c = (a+b)/2$. Then $a < c < b$, and c is a rational number because $c = (mp + nq)/(2qm)$. Then consider $d = (a+c)/2$ and $f = (c+b)/2$. These two numbers are both rational (write them explicitly in terms of m, n, p, q), and $a < d < c < f < b$ (again use Example 3 in the Equivalence Theorems section).

8. Consider $k=-1$. Then $2^{-1} > 4^{-1}$.

Uniqueness Theorems

1. For completeness sake, we must prove that: (a) the polynomial $p(x)$ has a solution, and (b) the zero is unique.
   a. *Existence.* Find the value(s) of the variable x for which $p(x)=0$. To do so, we have to solve the equation $x-b=0$. Using the properties of real numbers we obtain $x=b$.

b. *Uniqueness.* We can prove this in at least three ways:

i. We can use the result stating that a polynomial of degree n has at most n solutions. Therefore, a polynomial of degree 1 has one solution. Because we found it, it must be the only one.

ii. The solution is unique because of the algebraic process used to find it.

iii. We could assume that the number t is another zero of the polynomial $p(x)$. Thus, $p(t)=0$. Because $p(b)=0$ (from part a), we have $p(t)=p(b)$. This implies that $t-b=b-b$. Adding b to both sides of the equation yields $t=b$. This is the same solution we found in part a. Thus, the solution is unique.

2. In this case, we have to prove the existence of the solution of the equation $\cos\theta=\theta$ in the interval $[0,\pi]$. One way of achieving this goal is to graph the functions $f(x)=\cos x$ and $g(x)=x$. If the two graphs have only one intersection point in the interval $[0,\pi]$, the proof is complete. In this case, the graph can be used as a proof because we are interested in the situation on a *finite* interval; therefore, the graph shows all the possibilities. Another way to prove the statement is to graph the function $h(x)=(\cos x)/x$ and to show that there is only one value of x corresponding to $y=1$. (Be careful: This function is not defined at $x=0$. When $x=0$, $\cos 0=1$, so $\cos 0\neq 0$.)

3. We can start by finding a solution for the given equation and then prove that it is unique. Through algebraic manipulation, we obtain $x=\sqrt[3]{b}$ as a solution. Because it is possible to evaluate the third root of any real number, this expression is well defined for any value of b. Let y be another solution of the same equation. Then $x^3-b=0=y^3-b$. This implies that $x^3-y^3=0$. Using factorization techniques for the difference of two powers, this equation can be rewritten as $(x-y)(x^2+xy+y^2)=0$. This product will equal zero only if either $x-y=0$ or $x^2+xy+y^2=0$. The factor x^2+xy+y^2 is never equal to zero; it is irreducible (you can try to solve for one variable in terms of the other using the quadratic formula and check the sign of the discriminant.) Thus, the only possibility is that $x-y=0$. This implies that $x=y$ and the two solutions do indeed coincide. Therefore, the solution is unique.

4. A second-degree polynomial can be written as $P(x)=ax^2+bx+c$, where a, b, and c are real numbers, and $a\neq 0$. We will use the conditions given in the statement to find a, b, and c.

$$P(0)=a(0)^2+b(0)+c=c$$
$$P(1)=a(1)^2+b(1)+c=a+b+c$$
$$P(-1)=a(-1)^2+b(-1)+c=a-b+c.$$

Thus,

$$\begin{aligned} c &= -1 \\ a+b+c &= 3 \\ a-b+c &= 2. \end{aligned}$$

We can simplify this system of three equations and obtain

$$\begin{aligned} c &= -1 \\ a+b &= 4 \\ a-b &= 3. \end{aligned}$$

So, $a = 7/2$, $b = 1/2$, and $c = -1$. Thus, the polynomial that satisfies the given requirements is:

$$P(x) = \frac{7}{2}x^2 + \frac{1}{2}x - 1.$$

We have just proved that a polynomial satisfying the given requirements exists. Is this polynomial unique? The values of a, b, and c we obtained are the only solutions to the equations generated by the three conditions given in the statement, as we can see from the calculations performed to obtain them. (You might want to consult a linear algebra book for a more theoretical proof of this statement.) Therefore, the polynomial we obtained is the only one that satisfies the requirements.

5. To find the coordinates of the intersection points, set $f(x) = g(x)$. Then $x^3 = -x^2 - 2x$. This equation is equivalent to $x(x^2 + x + 2) = 0$. This product is zero if either $x = 0$ or $x^2 + x + 2 = 0$. The quadratic equation has no solution because its discriminant is negative. Thus, the equation $f(x) = g(x)$ has a unique solution: $x = 0$. The corresponding value of the y-coordinate is $y = 0$. Thus, the two graphs have a unique intersection point—namely, (0,0).

Equality of Sets

1. *First part:* $(A \cup B) \cup (A \cup C) \subseteq A \cup (B \cup C)$. Let $x \in (A \cup B) \cup (A \cup C)$. Then either $x \in (A \cup B)$ or $x \in (A \cup C)$. Thus, either ($x \in A$ or $x \in B$) or ($x \in A$ or $x \in C$). If we eliminate the redundant part of this sentence (the repeated information), we can rewrite it as $x \in A$ or $x \in B$ or $x \in C$. This implies that either $x \in A$ or $x \in (B \cup C)$. Thus, $x \in A \cup (B \cup C)$.

Second part: $A \cup (B \cup C) \subseteq (A \cup B) \cup (A \cup C)$. Let $x \in A \cup (B \cup C)$. Then, either $x \in A$ or $x \in (B \cup C)$. This implies $x \in A$ or $x \in B$ or $x \in C$. Therefore, either ($x \in A$ or $x \in B$) or ($x \in A$ or $x \in C$). Thus, we can conclude that $x \in (A \cup B) \cup (A \cup C)$.

2. *First part:* $A \subseteq B$. Let $x \in A$. Then x is a multiple of 2 and of 3. Therefore, $x = 2n$ with n integer number. Because x is divisible by 3 as well, while 2 is not, we can conclude that n is divisible by 3. So $x = 2n = 2(3m) = 6m$ with m integer number. Therefore, x is divisible by 6. Then, $x \in B$.

   Second part: $B \subseteq A$. Let $x \in B$. Then x is a multiple of 6. Thus, we can write $x = 6t$ for some integer number t. Then x is divisible by 2 and 3, because 6 is divisible by 2 and 3. So, $x \in A$.

3. *First part:* $(A \cup B)' \subseteq A' \cap B'$. Let $x \in (A \cup B)'$. This implies that $x \notin (A \cup B)$. Therefore, $x \notin A$ and $x \notin B$ (because if x belonged to either A or B, then it would belong to their union). Thus, $x \in A'$ and $x \in B'$. This implies that $x \in (A' \cap B')$.

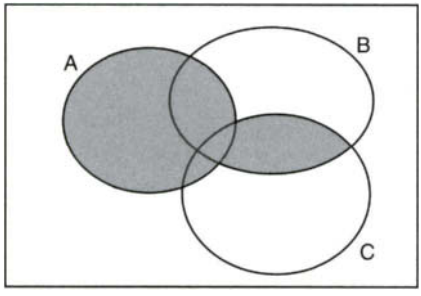
   Second part: $A' \cap B' \subseteq (A \cup B)'$. Let $x \in (A' \cap B')$. Then $x \in A'$ and $x \in B'$. Therefore, $x \notin A$ and $x \notin B$. This implies that $x \notin (A \cup B)$. So, we can conclude that $x \in (A \cup B)'$.

4. *First part:* $(A \cap B) \cap C \subseteq A \cap (B \cap C)$. Let $x \in (A \cap B) \cap C$. Thus, $x \in (A \cap B)$ and $x \in C$. This implies that $x \in A$ and $x \in B$ and $x \in C$. Then, $x \in A$ and $x \in (B \cap C)$. Therefore, $x \in A \cap (B \cap C)$.

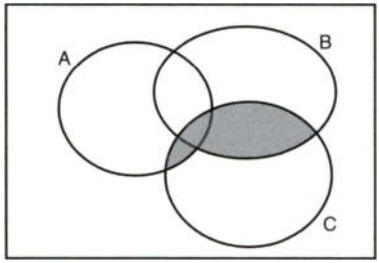
   Second part: $A \cap (B \cap C) \subseteq (A \cap B) \cap C$. Let $x \in A \cap (B \cap C)$. Then $x \in A$ and $x \in (B \cap C)$. This implies that $x \in A$ and $x \in B$ and $x \in C$. Thus, $x \in (A \cap B)$ and $x \in C$. Therefore, $x \in (A \cap B) \cap C$.

5. The two sets are not equal. The number 144 is in A, as $144 = 16 \times 9$, and $144 = 36 \times 4$, but 144 is not in B. So, $A \not\subset B$.

6. a. One can use a Venn diagram to get a better grasp of the sets involved. The following is a representation of the set $A \cup (B \cap C)$.



The following is a representation of the set $(A \cup B) \cap C$.



The equality does not seem to be true in general, as it contradicts the distributive law of union with respect to intersection. Let us look for a counterexample. If $A = \{1\}$, $B = \{2\}$, and $C = \{2,3\}$, then $A \cup (B \cap C) = A \cup \{2\} = \{1,2\}$ and $(A \cup B) \cap C = \{1,2\} \cap C = \{2\}$. Therefore, the two sets are not equal in general.

b. This equality seems to be an extension of one of De Morgan's laws. Let's try to prove it. The element x belongs to $(A \cap B \cap C)'$ if and only if $x \notin (A \cap B \cap C)$. This happens if and only if $x \notin A$ or $x \notin B$ or $x \notin C$. This is equivalent to saying that $x \in A'$ or $x \in B'$ or $x \in C'$. This happens if and only if $x \in A' \cup B' \cup C'$.

7. The element (x_0, y_0) belongs to A if and only if $y_0 = x_0^2 - 1$. This equality is equivalent to the equality:

$$y_0 = (x_0^2 - 1)\frac{x_0^2 + 1}{x_0^2 + 1}$$

because $x_0^2 + 1 \neq 0$. Therefore, $y_0 = x_0^2 - 1$ if and only if $(y_0 = (x_0^4 - 1)/(x_0^2 + 1))$. This means that $(x_0, y_0) \in A$ if and only if $(x_0, y_0) \in B$. Thus, the two sets are equal.

8. a. The base case has been proved in Example 5. Let $A = A_1$ and $B = A_2$.

b. *Inductive hypothesis:* Assume that for some $n \geq 3$ the equality $(A_1 \cap A_2 \cap \ldots \cap A_n)' = A_1' \cup A_2' \cup \ldots \cup A_n'$ holds true.

c. We have to prove that $(A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1})' = A_1' \cup A_2' \cup \ldots \cup A_n' \cup A_{n+1}'$. Using the associative property of intersection (see Exercise 4 in this section), we have:

$$(A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}) = [(A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}].$$

Therefore

$$(A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1})' = [(A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}]'.$$

Using the fact that we are now dealing with two sets (namely the set in parentheses and A_{n+1}), we have

$$\begin{aligned}(A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1})' &= [(A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}]' \\ &= (A_1 \cap A_2 \cap \ldots \cap A_n)' \cup A'_{n+1}.\end{aligned}$$

The inductive hypothesis and the associative property of intersection yield

$$\begin{aligned}(A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1})' &= [(A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}]' \\ &= (A_1 \cap A_2 \cap \ldots \cap A_n)' \cup A'_{n+1} \\ &= (A'_1 \cup A'_2 \cup \ldots \cup A'_n) \cup A'_{n+1} \\ &= A'_1 \cup A'_2 \cup \ldots \cup A'_n \cup A'_{n+1}.\end{aligned}$$

Therefore, by the principle of mathematical induction, the equality is true for all $n \geq 2$.

9. a. The smallest value of n we can use is 0. If a set has 0 elements it is the empty set. The only subset of the empty set is itself. Therefore, a set with 0 elements has 1 subset, and $1 = 2^0$. So the statement is true for $n = 0$.

   b. Assume that if $A = \{x_1, x_2, \ldots, x_n\}$, then A has 2^n subsets, $A_1, A_2, A_3, \ldots, A_{2^n}$.

   c. Let $B = \{b_1, b_2, \ldots, b_n, b_{n+1}\}$ be a set with $n+1$ elements. Does B have 2^{n+1} subsets? We can write $B = \{b_1, b_2, \ldots, b_n\} \cup \{b_{n+1}\}$. The set $\{b_1, b_2, \ldots, b_n\}$ has n elements; therefore, it has 2^n subsets, $B_1, B_2, B_3, \ldots, B_{2^n}$. None of these subsets includes b_{n+1}. Every one of these subsets is a subset of B as well, and it uses only the first n elements of B. Thus, we can construct more sets of B using b_{n+1}. Thus, we have the following subsets of B: $B_1, B_2, B_3, \ldots, B_{2^n}$, $B_1 \cup \{b_{n+1}\}, B_2 \cup \{b_{n+1}\}, B_3 \cup \{b_{n+1}\}, \ldots, B_{2^n} \cup \{b_{n+1}\}$. Therefore, B has $2^n + 2^n = 2^{n+1}$ subsets. Thus, by the principle of mathematical induction, the statement is true for all $n \geq 0$.

Equality of Numbers

1. *Part 1.* Let us assume that $(x-y)^5 + (x-y)^3 = 0$. Will this imply that $x = y$? Using the distributive property, we can rewrite this equality as $(x-y)^3[(x-y)^2 + 1] = 0$. The product of several factors is equal to zero if and only if at least one of the factors equals zero. Therefore, either $(x-y)^3 = 0$ or $[(x-y)^2 + 1] = 0$. The first equality implies

$x - y = 0$, or $x = y$. The proof is not complete because we still have to prove that this is the only possible conclusion. The second equality can be rewritten as $(x - y)^2 = -1$. Because $(x - y)^2$ is always non-negative, this equality will never be true. Therefore, the product $(x - y)^3[(x - y)^2 + 1]$ is equal to zero only if $(x - y)^3 = 0$; that is, when $x = y$.

Part 2. We have to prove that if $x = y$, then $(x - y)^5 + (x - y)^3 = 0$. This is quite easy to do; indeed, in this case $x - y = 0$.

2. *Part 1.* These two sequences are equal if $x^n = y^n$ for all $n \geq 2$. Because $x^2 = y^2$, we obtain $(x - y)(x + y) = 0$. Therefore, we have two possible conclusions: either $x = y$ or $x = -y$. We can only accept the conclusion $x = y$. Indeed, if $x = -y$, then $x^3 = -y^3$. But $x^3 = y^3$ by hypothesis.

   Part 2. The converse of this statement is trivial.

3. By definition, a divides b if the division of b by a yields a counting number and zero remainder. Therefore, we can write $b/a = q$, with q counting number. Similarly $c/b = t$, where t is a counting number, and $a/c = s$, where s is a counting number. These three equalities can be rewritten as $b = aq$, $c = bt$, $a = cs$. If we use all of them, we obtain $b = (cs)q = c(sq) = (bt)(sq) = b(tsq)$ (#). Therefore, $b = b(tsq)$. Because $b \neq 0$ (since 0 is not a counting number), we obtain $1 = tsq$. As t, s, and q are all counting numbers and are larger than or equal to 1, tsq can equal 1 if and only if $t = 1$, $s = 1$, and $q = 1$. If we use this result in (#), we obtain $b = a$ $c = b$ $a = c$. Therefore, $a = b = c$.

4. Let a, b, and c be three counting numbers. Set $d = GCD(ac, bc)$ and $e = GCD(a, b)$. We want to prove that $d = ce$.

   Part 1. $d \geq ce$. Because $e = GCD(a, b)$, we can write $a = ke$ and $b = se$ with k and s relatively prime. Multiplying both equalities by c, we obtain $ac = k(ce)$ and $bc = s(ce)$. This proves that ce is a common divisor of ac and bc. But d is the *greatest* common divisor. Thus, $d \geq ce$.

   Part 2. $d \leq ce$. Because e is the greatest common divisor of a and b, we can write $a = ke$ and $b = se$, where k and s are relatively prime. So, multiplying by c, we obtain $ac = k(ce)$ and $bc = s(ce)$, where k and s are relatively prime. Then all the common factors of ac and bc are in ce. Thus, ce is larger than any other common divisor. So $d \leq ce$. From the two parts of this proof we can conclude that $d = ce$.

5. By hypothesis $(a/b)^m = n$, where n is an integer. Thus, $a^m = b^m n$ or $a^m = b(b^{m-1}n)$. This means that b divides a^m. Because a and b are relatively prime, their greatest common divisor is 1. Therefore, b and a^m cannot have any common factors other than 1. Thus, $b = 1$.

COMPOSITE STATEMENTS

1. We can rewrite the equation $x^2 = y^2$ as $x^2 - y^2 = 0$. We can factor the difference of two squares and rewrite the equation as $(x - y)(x + y) = 0$. This equality implies that either $x - y = 0$ or $x + y = 0$. If both x and y are equal to zero, both equalities are trivially true and $x = y$. Therefore, we will assume that $x \neq 0$ and $y \neq 0$. From the previous equalities we obtain that either $x = y$ or $x = -y$. If $x = -y$, because neither x nor y is equal to zero one of them would be a positive number and the other would be negative. But the second part of the hypothesis states that the two numbers are nonnegative. So, we have to reject this case.

   Thus, the only conclusion we can accept is that $x = y$.

2. The function f is even. So $f(x) = f(-x)$ for all x in its domain. Because the function f is odd as well, $f(-x) = -f(x)$ for all x in its domain. Combining these two hypotheses yields $f(x) = f(-x) = -f(x)$.

   Thus, $2f(x) = 0$ for all x in the domain of f. This means that $f(x) = 0$ for all x in the domain of the function.

3. Let n be a multiple of 3, and assume that n is not odd. Then it is even, and therefore divisible by 2. Because n is divisible by 2 and by 3, it is divisible by 6. (Even if $n = 6$ the statement is true as 6 is divisible by 6.) Thus, the two choices listed in the conclusion are the only two possible ones. The statement is therefore true.

4. Let us assume that $x \neq y$. We can rewrite the equality $x^4 = y^4$ as $(x - y)(x + y)(x^2 + y^2) = 0$. Therefore, $x - y = 0$ or $x + y = 0$ or $x^2 + y^2 = 0$. The first equality implies $x = y$, but we have excluded this possibility. The second equality implies that $x = -y$. The statement is true if there are no more possible choices but this. The last equality is possible if and only if $x = y = 0$. Because we are working under the hypothesis that $x \neq y$, we cannot accept this conclusion. Therefore, if $x \neq y$ the only possibility left is that $x = -y$.

5. We will assume that A is nonempty. The set $A - B$ is empty by hypothesis. By definition of the set $A - B$, this means that there is no element of A that does not belong to B. Then all the elements of A belong to B. Therefore, $A \subseteq B$.

6. *Case 1.* Let $A = \emptyset$. By definition of union, $A \cup B = \{x | x \in A \text{ or } x \in B\}$. Because the condition $x \in A$ is always false (*i.e.*, there exists no $x \in A$), then $A \cup B = \{x | x \in A \text{ or } x \in B\} = \{x | x \in B\} = B$.
   Case 2. Let $A \subseteq B$. By definition of union, $A \cup B = \{x | x \in A \text{ or } x \in B\}$. By hypothesis, $A \subseteq B$. Therefore, $x \in A$ implies $x \in B$. Thus, the

condition $x \in A$ is redundant when selecting the elements in the union. Indeed, every element of A will be in the union because it is an element of B. Thus, $A \cup B = \{x | x \in A \text{ or } x \in B\} = \{x | x \in B\} = B$.

Limits

1. Let $\varepsilon > 0$ be given. We need to prove the existence of a number $\delta > 0$ such that if $|x - 1| < \delta$, then $|(3x^2 + 2) - 5| < \varepsilon$. Because $|(3x^2 + 2) - 5| = |3x^2 - 3| = 3|x^2 - 1| = 3|x + 1||x - 1|$, we need to estimate how large the value of the factor $|x + 1|$ can be. Because x has to be in an interval centered at 1, we can choose $0 < x < 2$ (*i.e.*, an interval of radius 1). (This is a completely arbitrary choice. Other choices will work as well; they will just yield different results for $\delta > 0$. For example, check what would happen when one uses $-1 < x < 3$.) Therefore, $1 < x + 1 < 3$; that is, $1 < |x + 1| < 3$. Thus,

$$|(3x^2 + 2) - 5| = 3|x + 1||x - 1| < 3 \times 3|x - 1| = 9|x - 1|.$$

   Because we want $9|x - 1| < \varepsilon$, we need $|x - 1| < \varepsilon/9$. In conclusion, choose $\delta = \text{minimum}\{1, \varepsilon/9\}$. Because $\varepsilon > 0$, we have $\delta > 0$. Remember that it is possible to check that if $|x - 1| < \delta$, where $\delta > 0$ is the number we determined, then $|(3x^2 + 2) - 5| < \varepsilon$.

2. Let $\varepsilon > 0$ be given. We need to prove the existence of a number $\delta > 0$ such that if $|x - 2| < \delta$, then $(|1/(x^2 + 1) - (1/5)| < \varepsilon)$.

   Algebraic steps yield:

$$\left|\frac{1}{x^2 + 1} - \frac{1}{5}\right| = \left|\frac{5 - (x^2 + 1)}{5(x^2 + 1)}\right| = \left|\frac{4 - x^2}{5(x^2 + 1)}\right| = \frac{|4 - x^2|}{|5(x^2 + 1)|}.$$

   The quantity $5(x^2 + 1)$ is always positive (because $x^2 + 1$ is always positive). So, $5(x^2 + 1) = |5(x^2 + 1)|$. Moreover, we need to keep in mind that we want to estimate the value of $|x - 2|$. Thus, we can consider the following equality $|4 - x^2| = |x^2 - 4| = |x + 2||x - 2|$. Therefore, $\left|\frac{1}{x^2+1} - \frac{1}{5}\right| = \frac{|(x+2)|}{5(x^2+1)}|x - 2|$ and we need to estimate the largest value that the fraction $\frac{|x+2|}{5(x^2+1)}$ can have for x in an interval centered at 2. Because x has to be in an interval centered at 2, we can choose $0.5 < x < 3.5$ (i.e., an interval of radius 1.5). (This is a completely arbitrary choice. Other choices will work as well; they will just yield different results for $\delta > 0$. For example, check what would happen when one uses $1 < x < 3$ or $0 < x < 4$.) Therefore, $2.5 < x + 2 < 5.5$; that is, $2.5 < |x + 2| < 5.5$. Moreover $0.25 < x^2 < 12.25$.

Thus, $1.25 < x^2 + 1 < 13.25$, and $6.25 < 5(x^2 + 1) < 66.25$; that is, $(1/66.25) < (1/5(x^2 + 1)) < (1/6.25)$. Combining these estimates with the ones for $|x + 2|$ yields $\frac{|(x+2)|}{5(x^2+1)} < \frac{5.5}{6.25} = \frac{22}{25}$. Thus, $\left|\frac{1}{(x^2+1)} - \frac{1}{5}\right| = \frac{|(x+2)|}{5(x^2+1)}|x - 2| < \frac{22}{25}|x - 2|$. To have this expression be smaller than ε, one needs $|x - 2| < \frac{25}{22}\varepsilon$. Therefore, let $\delta = \text{minimum}\{1.5, \frac{25}{22}\varepsilon\}$.

3. Let $\varepsilon > 0$ be given. We need to prove the existence of a number $\delta > 0$ such that if $|x - 1| < \delta$, then $\left|\frac{x^3-1}{x^2-1} - \frac{3}{2}\right| < \varepsilon$. Note that the function $(x^3 - 1)/(x^2 - 1)$ is undefined at 1 and -1. It can be rewritten as $(x^3 - 1)/(x^2 - 1) = (x - 1)(x^2 + x + 1)/(x - 1)(x + 1) = (x^2 + x + 1)/(x + 1)$ when $x \neq 1$ and $x \neq -1$. Thus, additional algebraic steps yield $\left|\frac{x^3-1}{x^2-1} - \frac{3}{2}\right| = \left|\frac{x^2+x+1}{x+1} - \frac{3}{2}\right| = \left|\frac{2x^2+2x+2-3x-3}{2(x+1)}\right| = \left|\frac{2x^2-x-1}{2(x+1)}\right|$. Because $2x^2 - x - 1 = (2x + 1)(x - 1)$, we have $\left|\frac{x^3-1}{x^2-1} - \frac{3}{2}\right| = \frac{|2x+1|}{2|x+1|}|x - 1|$. Therefore, an estimate for $\frac{|2x+1|}{2|x+1|}$ is needed, knowing that x is in an interval centered at 1. Because x has to be in an interval centered at 1, we can choose $0 < x < 2$ (*i.e.*, an interval of radius 1). (This is a completely arbitrary choice. Other choices will work as well; they will just yield different results for $\delta > 0$. For example, check what would happen when one uses $-1 < x < 3$.) Therefore, $1 < x + 1 < 3$; that is, $1 < |x + 1| < 3$. Then $\frac{1}{3} < \frac{1}{|x+1|} < 1$, and $\frac{1}{6} < \frac{1}{2|x+1|} < \frac{1}{2}$. Let us now consider the factor $|2x + 1|$. Because $1 < 2x + 1 < 5$, $1 < |2x + 1| < 5$. Thus, $\frac{|2x+1|}{2|x+1|} < \frac{5}{2}$, and $\left|\frac{x^3-1}{x^2-1} - \frac{3}{2}\right| < \frac{5}{2}|x - 1|$. For this quantity to be smaller than the given $\varepsilon > 0$, one needs $|x - 1| < \frac{2}{5}\varepsilon$. Therefore, let $\delta = \text{minimum}\{1, \frac{2}{5}\varepsilon\}$.

4. Let $\delta = 0.9$. Then $|x - 2| < \delta = 0.9$. This implies $-0.9 < x - 2 < 0.9$; that is, $1.1 < x < 2.9$. Therefore, $3.3 < 3x < 8.7$. Subtracting 6 yields $-2.7 < (3x - 5) - 1 < 2.7$. So, $|(3x - 5) - 1| < 2.7 < 4.5$.

5. Assume that $|x - 2| < \delta \leq 1.5$. Then $-1.5 \leq -\delta < x - 2 < \delta \leq 1.5$. Adding 2 to all the parts of the inequality yields $0.5 \leq 2 - \delta < x < 2 + \delta \leq 3.5$. In turn, multiplying these inequalities by 3 implies that $1.5 \leq 3(2 - \delta) < 3x < 3(2 + \delta) \leq 10.5$. Subtracting 6 yields $1.5 - 6 < 3x - 6 < 10.5 - 6$ (*i.e.*, $-4.5 < 3x - 6 < 4.5$). Therefore, $|(3x - 5) - 1| < 4.5$ whenever $|x - 2| < \delta \leq 1.5$.

6. To prove that $\lim_{n\to\infty} \frac{1}{3n+1} = 0$ means proving that for every given $\varepsilon > 0$ there exists an N such that $\left|\frac{1}{3n+1} - 0\right| < \varepsilon$ for all $n > N$. Because $\left|\frac{1}{3n+1} - 0\right| = \left|\frac{1}{3n+1}\right| = (1/(3n + 1))$, we need to find values of n that satisfy the inequality $(1/(3n + 1)) < \varepsilon$. This yields $3n + 1 > (1/\varepsilon)$, or $n > (1/3)((1/\varepsilon) - 1)$. Thus, let $N = (1 - \varepsilon)/(3\varepsilon)$.

7. To prove that $\lim_{n\to\infty}\frac{1}{n^2+1}=0$ means proving that for every given $\varepsilon>0$ there exists an N such that $\left|\frac{1}{n^2+1}-0\right|<\varepsilon$ for all $n>N$. Because $\left|\frac{1}{n^2+1}-0\right|=\left|\frac{1}{n^2+1}\right|=\frac{1}{n^2+1}$, we need to find values of n that satisfy the inequality $((1/(n^2+1))<\varepsilon$. This yields $n^2+1>(1/\varepsilon)$, or $n^2>(1/\varepsilon)-1$. This is a second-degree inequality, and its solution set is $n<-\sqrt{1/\varepsilon-1}$ or $n>\sqrt{1/\varepsilon-1}$. Because $n\geq 1$, we will only choose $n>N=\sqrt{1/\varepsilon-1}$. Note that N is a real number only when $1\geq\varepsilon>0$. In any case, when $\varepsilon>1$, the inequality $\left|\frac{1}{n^2+1}-0\right|<\varepsilon$ will be true for all values of n. Indeed $(1/(n^2+1))<1$ for all values of n.

8. To prove that $\lim_{n\to\infty}\frac{5n+1}{3n-2}=\frac{5}{3}$ means proving that for every given $\varepsilon>0$ there exists an N such that $\left|\frac{5n+1}{3n-2}-\frac{5}{3}\right|<\varepsilon$ for all $n>N$. Simplification of the difference between the terms of the series and 5/3 yields $\left|\frac{5n+1}{3n-2}-\frac{5}{3}\right|=\left|\frac{3(5n+1)-5(3n-2)}{3(3n-2)}\right|=\left|\frac{13}{3(3n-2)}\right|=\frac{13}{3|3n-2|}$. Because $n\geq 1$, the expression $3n-2>0$. Thus, $\left|\frac{5n+1}{3n-2}-\frac{5}{3}\right|=\frac{13}{3(3n-2)}$, and we need to find values of n that satisfy the inequality $(13/(3(3n-2)))<\varepsilon$. Therefore, we obtain $n>N=\frac{1}{3}\left(\frac{13}{3\varepsilon}+2\right)$.

9. Consider again the calculation performed in Example 4:

$$\left|\frac{2n-1}{n+1}-2\right|=\left|\frac{(2n-1)-2(n+1)}{n+1}\right|=\left|\frac{-3}{n+1}\right|=\frac{|-3|}{|n+1|}=\frac{3}{n+1}.$$

If $n>M_{4/5}=16$, then $n+1>17$; therefore, $\left|\frac{2n-1}{n+1}-2\right|=\frac{3}{n+1}<\frac{3}{17}<\frac{4}{5}$.

10. Let us start by observing that

$$\left|\frac{n+1}{n^2}-0\right|=\frac{n+1}{n^2}=\frac{1}{n}+\frac{1}{n^2}=\frac{1}{n}\left(1+\frac{1}{n}\right).$$

Case 1. $n>N_\varepsilon=\frac{1}{2\varepsilon}\left(1+\sqrt{1+4\varepsilon}\right)$. Because $n>N_\varepsilon=\frac{1}{2\varepsilon}\left(1+\sqrt{1+4\varepsilon}\right)$, $((1/n)<(2\varepsilon)/(1+\sqrt{1+4\varepsilon}))$. Moreover note that

$$\left(1+\sqrt{1+4\varepsilon}\right)^2=1+2\sqrt{1+4\varepsilon}+1+4\varepsilon=2+4\varepsilon+2\sqrt{1+4\varepsilon}.$$

Therefore, $\left|\frac{n+1}{n^2}-0\right|=\frac{1}{n}\left(1+\frac{1}{n}\right)<\frac{2\varepsilon}{1+\sqrt{1+4\varepsilon}}\left(1+\frac{2\varepsilon}{1+\sqrt{1+4\varepsilon}}\right)$. Thus,

$$\left|\frac{n+1}{n^2}-0\right|<\frac{2\varepsilon}{1+\sqrt{1+4\varepsilon}}\left(\frac{1+\sqrt{1+4\varepsilon}+2\varepsilon}{1+\sqrt{1+4\varepsilon}}\right)$$
$$=\frac{2\varepsilon}{\left(1+\sqrt{1+4\varepsilon}\right)^2}\frac{1}{2}(2+4\varepsilon+\sqrt{1+4\varepsilon})=\varepsilon.$$

Case 2. $n > M_\varepsilon = ((1+\varepsilon)/\varepsilon)$. Because $\frac{1}{n} < \frac{\varepsilon}{1+\varepsilon}$, we have $\left|\frac{n+1}{n^2} - 0\right| = \frac{1}{n}\left(1+\frac{1}{n}\right) < \frac{\varepsilon}{1+\varepsilon}\left(1+\frac{\varepsilon}{1+\varepsilon}\right) = \frac{\varepsilon}{(1+\varepsilon)^2}(1+2\varepsilon)$.
Moreover, $(1+2\varepsilon) < (1+2\varepsilon+\varepsilon^2) = (1+\varepsilon)^2$.
Therefore, $\left|\frac{n+1}{n^2} - 0\right| < \frac{\varepsilon}{(1+\varepsilon)^2}(1+2\varepsilon) < \frac{\varepsilon}{(1+\varepsilon)^2}(1+2\varepsilon)^2 = \varepsilon$.

SOLUTIONS FOR THE REVIEW EXERCISES

1. We are assuming that the two points, P and Q are distinct. Therefore, the values of their x-coordinates or the values of their y-coordinates are different; that is, either $x_1 \neq x_2$ or $y_1 \neq y_2$. We will assume that $x_1 \neq x_2$ (geometrically, this means that the points are not on the same vertical line). Because this implies that $x_1 - x_2 \neq 0$, we know that $(x_1 - x_2)^2 > 0$. The quantity $(y_1 - y_2)^2$ is always nonnegative (because it is the second power of a real number). Therefore, $(x_1 - x_2)^2 + (y_1 - y_2)^2 \geq (x_1 - x_2)^2 > 0$. This implies that d itself is a positive number. We have proved that the given statement is true under the assumption that $x_1 \neq x_2$. Similarly, we can prove that the statement is true under the assumption $y_1 \neq y_2$ (geometrically, this means that the points are not on the same horizontal line). The part of the proof for the y-coordinates is not a "must" if one observes that the formula used to evaluate the distance is symmetric with respect to the x- and y-coordinates (that means that we could switch the two coordinates and the formula would not change); therefore whatever was proved for one of the coordinates is true for the other one as well. There is a third part of the proof that is not needed, but we want to mention it for completeness sake. It is possible to assume that $x_1 \neq x_2$ and $y_1 \neq y_2$ at the same time. The given statement is still true in this case, because we have proved that it holds true when only one pair of coordinates has different values, which is a much weaker assumption than this last one.

2. Let a be any real number and a' be an opposite of a. Then

$$a + a' = a' + a = 0 \tag{1}$$

Let b be another number such that

$$a + b = b + a = 0 \tag{2}$$

(Because we have two distinct numbers acting similarly on the number a, we should wonder how they interact with each other. The answer to this question is not evident because we only know Equations (1) and (2), and both sets of equalities involve a. Then let us try to construct

some algebraic expression that uses all the numbers we are considering, namely a, a', and b. It makes sense to keep using addition as this is the only operation used to define the opposite of a number. These are some of the reasons for trying to start from $a+a'+b$. Which conclusion can we reach?) If we use the associative property of addition and Equation (1), we obtain $a+a'+b=(a+a')+b=0+b=b$. If we use the associative and commutative property of addition and Equation (2), we obtain $a+a'+b=a'+(a+b)=a'+0=a'$. Therefore, we have $b=a+a'+b=a'$. Thus, $b=a'$, and the opposite of a is unique.

3. a. Proof by induction:

   i. The smallest positive integer number to use is 1. Because $\ln 1=0$, it is true that $\ln 1<1$.

   ii. Let us assume that the inequality is true for n. Thus, $\ln n<n$.

   iii. We have to prove that $\ln(n+1)<(n+1)$.

   (*Remember*: $\ln(n+1)\neq \ln n+\ln 1$.)

   We have to try to use what we know about n and $n+1$. One possible relation is:

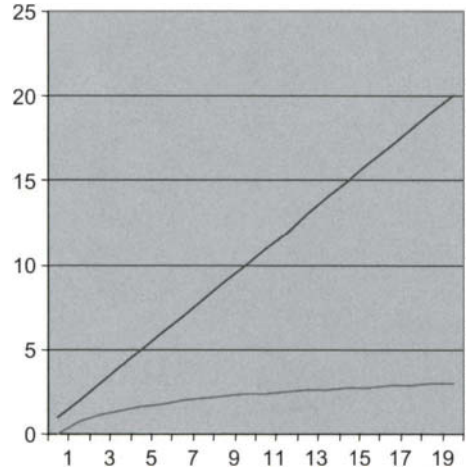
$$n+1=\frac{n+1}{n}n.$$

   Therefore, using the properties of the natural logarithm, we have:

$$\begin{aligned}\ln(n+1)&=\ln\left(\frac{n+1}{n}n\right)\\&=\ln\left(\frac{n+1}{n}\right)+\ln n\\&=\ln(1+1/n)+\ln n.\end{aligned}$$

   By the inductive hypothesis, $\ln n<n$. Thus, we have $\ln(n+1)=\ln(1+1/n)+\ln n<\ln(1+1/n)+n$. To show that the conclusion is true, we need to prove that $\ln(1+1/n)\leq 1$. Because $n\geq 1$, $1/n\leq 1$. Therefore, $1+1/n\leq 1+1=2\leq e=2.72\ldots$. So $1+1/n\leq e$. The function natural logarithm is an increasing function; thus, the larger its input, the larger the corresponding output. Thus, $\ln(1+(1/n))\leq \ln e=1$. If we use this information in the chain of inequalities, we obtain $\ln(n+1)<\ln(1+1/n)+n\leq 1+n$, or, equivalently, $\ln(n+1)<n+1$. By the principle of mathematical induction the proof is now complete.

   b. By graphing—The straight line is the graph of $g(x)=x$, the other is the graph of $f(x)=\ln x$. From the graph we can say that it seems plausible that $\ln n<n$ for all positive integers.



c. Consider the function $h(x) = \ln x/x$ for $x \geq 1$. This function is never negative because $\ln x \geq 0$ for $x \geq 1$. Moreover, $h(1) = 0$. We can either graph the function or use the first derivative test to check whether the function is increasing or decreasing and to find its critical point(s).

$$h'(x) = \frac{x/x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

The function h has a critical point for the value of x such that $1 - \ln x = 0$. Because $\ln x = 1$ when $x = e$, this is the only critical value for the function h. We will use the second derivative test to decide what kind of critical point this is.

$$h''(x) = \frac{(-1/x)x^2 - 2x(1 - \ln x)}{x^4} = \frac{2\ln x - 3}{x^3}.$$

Therefore,

$$h''(e) = \frac{2\ln e - 3}{e^3} = \frac{2 - 3}{e^3} = -\frac{1}{e^3}.$$

This is a negative number. So the value $x = e$ corresponds to a local maximum of the function. Thus, $h(x) \leq h(e)$ for all $x \geq 1$; that is, $h(x) \leq h(e) = (\ln\ e)/e \approx 0.36\ldots$. Therefore, $h(x) < 1$ for all $x \geq 1$. Because $(\ln\ x)/x < 1$ for all $x \geq 1$, we can conclude that $\ln x < x$ for all $x \geq 1$. In particular, $\ln n < n$ for all positive integers, as they are just real numbers larger than or equal to 1.

d. We will consider the function $g(x) = \ln x - x$ for $x \geq 1$. We know that $g(1) = -1$. Let's study the behavior of this function using the first derivative test. Because $g'(x) = -1 + 1/x$, it follows that g' is always negative for $x > 1$ and that $g'(1) = 0$. Thus, $g(1) = -1$ is the maximum value of g. This implies that $g(x) \leq g(1) < 0$ for all $x \geq 1$. This proves that $\ln x < x$ for all $x \geq 1$. In particular, $\ln n < n$ for all positive integers, as they are just real numbers larger than or equal to 1.

4. a. To prove that the two sets are equal, we need to prove that they have the same elements.

Part 1. $A \subseteq B$. Let x be an element of A. Then x is a multiple of 15; that is, $x = 15n$ for some integer number n. Therefore, x is a multiple of 5, because $x = 5(3n)$, and x is a multiple of 3, because $x = 3(5n)$. This means that x is an element of B.

Part 2. $B \subseteq A$. Let y be an element of B. Then y is a multiple of 5 and a multiple of 3. Thus, we can write $y = 3p$ for some integer number p and $y = 5s$ for some integer number s. Therefore, $y = 3p = 5s$. Because 3 is not divisible by 5, p must be divisible by 5. So, $p = 5q$ for some integer number q. Thus, $y = 3p = 3(5q) = 15q$ for some integer number q. This proves that y is a multiple of 15 and it belongs to A.The two parts of the proof imply that $A = B$.

b. *Part 1.* $A \subseteq B$. See part 1 above.
Part 2. $B \subseteq A$. This inclusion is not true. Consider the number 6. It is a multiple of 3; therefore, it belongs to B. But it is not an element of A, as it is not a multiple of 15. Therefore, the set A cannot contain the set B. Moreover, because of part 1, we know that A is contained in B. This means that A is a proper subset of B.

5. i. A solution exists. Because the number a is not equal to 0, it has a reciprocal, a^{-1}. Then we can multiply both sides of the equation $ax = b$ by $a^{-1}$to obtain $a^{-1}(ax) = a^{-1}b$. So, the solution is $x = a^{-1}b$.

ii. We can prove that the solution $t = a^{-1}b$ is unique in two ways: (1) The solution is unique because of the algebraic procedure used to find it and the fact that the reciprocal of a number is unique. (2) Let s be another solution of the equation. Then $as = b$ and $at = b$. This implies that $as = at$. If we multiply both sides of the equation by a^{-1}, we obtain $s = t$. This means that there is only one solution.

6. *Part 1.* If n^3 is an odd number, then n is an odd number. (Because n^3 is an odd number, we can write $n^3 = 2q + 1$ for some integer number q. To find any information about n we need to calculate the cubic root of $2q + 1$; however this is no easy task because there is no easy

formula to calculate the cubic root of a sum. Therefore, the fact that n^3 is an odd number does not seem to give us an effective starting point for the proof, and we could try to use the contrapositive of the statement to be proved.) Prove the statement: "If n is not an odd number, then n^3 is not an odd number." Because n is not an odd number, it must be even. Therefore, we can write $n=2p$ for some integer number p. Then $n^3 = (2p)^3 = 8p^3 = 2(4p^3)$. Because $4p^3$ is an integer number, the equality above proves that n^3 is an even number; that is, n^3 is not an odd number. Because the contrapositive of the original statement is true, the original statement is true.

Part 2. If n is an odd number, then n^3 is an odd number. Because n is an odd number, we can write $n=2q+1$ for some integer q. Therefore, $n^3 = 8q^3 + 12q^2 + 6q + 1 = 2(4q^3 + 6q^2 + 3q) + 1$. The number $t = 4q^3 + 6q^2 + 3q$ is an integer. Thus, we have $n^3 = 2t + 1$ with t integer number. This means that n^3 is an odd number. So the statement is true.

7. *Part 1.* Statement (a) implies statement (b). Suppose that the two inequalities in statement (a) hold true. We can combine them and we obtain:

$$\mathrm{a} \leq \mathrm{b} \leq \mathrm{a} \qquad (*)$$

Because the number a cannot be strictly smaller than itself, the chain of inequalities (*) can be true only if the two relations are equalities. Therefore, we have $a=b=a$. Because $a=b$, we conclude that $a-b=0$.

Part 2. Statement (b) implies statement (a). Because $a-b=0$, we know that a and b are indeed equal. Thus, the inequalities $a \leq b$ and $b \leq a$ are trivially true. The proof is now complete.

8. This is an existence and uniqueness theorem. Indeed the statement can be read as:

   a. Any nonzero number has a reciprocal. (This is an axiom.)

   b. Such reciprocal is unique. (This has to be proved.)

   We assume that there are at least two numbers, a^{-1} and s, with the properties $as = sa = 1$ and $aa^{-1} = a^{-1}a = 1$. We want to prove that $a^{-1} = s$. Therefore, we need to use the properties of these two numbers to compare them. We can obtain the following chain of equalities: $s = s1 = s(aa^{-1}) = (sa)a^{-1} = 1a^{-1} = a^{-1}$. Thus, $a^{-1} = s$. Therefore, a has a unique reciprocal.

9. Because the statement mentions "factors" and "division," we might consider p and q as products of their prime factors. Thus, $p = p_1 p_2 \ldots p_{r-1} p_r$ and $q = q_1 q_2 \ldots q_{s-1} q_s$. The numbers p_i need

not to be distinct. Similarly, the numbers q_j need not be distinct, but no p_i can equal any q_j. Using the factorization of p above and the properties of multiplication, we have $p^n = (p_1)^n(p_2)^n \ldots (p_{r-1})^n(p_r)^n$. Because the prime factors of p^n are the same as the prime factors of p, p^n and q have no common factors. Therefore, q cannot divide p^n.

10. *Step 1*: Is the formula true for $n = 1$, the smallest number we can use? When $n = 1$, we obtain

$$\frac{1}{1}\frac{1}{2} = \frac{1}{1+1}.$$

Therefore, the formula is true in this case.

Step 2. Assume that the formula is true for an arbitrary number n; that is,

$$\frac{1}{1}\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{4} + \cdots + \frac{1}{n}\frac{1}{n+1} = \frac{n}{n+1}.$$

Step 3. Show that the formula is true when we use it for the next integer number (namely, $n+1$). So, we need to prove that:

$$\frac{1}{1}\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{4} + \cdots + \frac{1}{n}\frac{1}{n+1} + \frac{1}{(n+1)}\frac{1}{(n+2)} = \frac{n+1}{(n+1)+1},$$

or, equivalently,

$$\frac{1}{1}\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{4} + \ldots + \frac{1}{n}\frac{1}{n+1} + \frac{1}{(n+1)}\frac{1}{(n+2)} = \frac{n+1}{n+2}.$$

We will manipulate the expression on the left-hand-side of the equation using first the associative property of addition, then the inductive hypothesis, and then some algebraic steps:

$$\begin{aligned}
&\frac{1}{1}\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{4} + \cdots\cdots + \frac{1}{n}\frac{1}{n+1} + \frac{1}{(n+1)}\frac{1}{(n+2)} \\
&= \left\{\frac{1}{1}\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{4} + \cdots\cdots + \frac{1}{n}\frac{1}{n+1}\right\} + \frac{1}{(n+1)}\frac{1}{(n+2)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}.
\end{aligned}$$

Because this is exactly the equality we were trying to prove, the formula is indeed true for all positive integer numbers by the principle of mathematical induction.

11. (The statement has only implicit hypotheses. Before proceeding we must be sure that we are familiar with the definition of rational numbers and their operations and properties. We can reformulate the statement in the following way: If q is a rational number, then $q \neq \sqrt{2}$. This statement is equivalent to: If q is a rational number, then $q^2 \neq 2$. Because we cannot directly check that the square of each rational number is not equal to 2, we will try to prove the contrapositive of the statement.) The statement to be proved is "If q is a rational number, then $q^2 \neq 2$." We will assume that there exists a rational number q such that $q^2 = 2$. Because q is a rational number, it can be written as $q = a/b$, where a and b are relatively prime integer numbers, $b \neq 0$, and $a \neq 0$ (because $q \neq 0$ as $0^2 \neq 2$). Because $q^2 = 2$, we have $\frac{a^2}{b^2} = 2$. This is equivalent to:

$$a^2 = 2b^2 \qquad (*)$$

Thus, a^2 is a multiple of 2. This implies that a is a multiple of 2. Therefore, $a = 2k$ for some integer number k. If we substitute into equation (*), we obtain $4k^2 = 2b^2$. Thus, $2k^2 = b^2$. This implies that b^2 is a multiple of 2. Therefore, b is a multiple of 2; that is, $b = 2s$ for some integer number s. Our calculations show that a and b have at least 2 as a common factor. However, by hypothesis, a and b are relatively prime integer numbers. Because it is impossible to find two numbers that satisfy both these conditions at the same time, we cannot find a rational number such that $q^2 = 2$.

12. Let $y = ax + b$ and $y = cx + d$ be the equations of the two lines. Because the lines are distinct by hypothesis, we know that either $a \neq c$ or $b \neq d$. The coordinates of the intersection point are the solutions of the system:

$$\begin{cases} y = ax + b \\ y = cx + d. \end{cases}$$

Therefore, we obtain $ax + b = cx + d$ or $(a - c)x = d - b$. If $a - c = 0$, the system has no solutions because $b \neq d$. (We can explain this result geometrically. The two lines have the same slope; therefore, they are parallel and distinct. Thus, they have no points in common.) If $a - c \neq 0$, we obtain the only solution, $x = \frac{d-b}{a-c}$. Therefore, the unique intersection point is the one having coordinates $\left(\frac{d-b}{a-c}, \frac{ad-bc}{a-c}\right)$. (This statement can be proved using its contrapositive as well. In this case, start by assuming that the two lines have two points in common, and use algebraic steps to obtain the conclusion that $a = c$ and $b = d$.)

13. (Because it impossible to check directly all negative numbers, we have to find a different way to prove that the statement is true. Try using the contrapositive.) Assume that there exists a negative number z whose reciprocal, z^{-1}, is not negative. By definition of reciprocal of a number, $z \times z^{-1} = 1$. By the rules of algebra, $z^{-1} \neq 0$. Therefore, z^{-1} must be a positive number; however, the product of a negative and a positive number is a negative number. This conclusion generates a contradiction because 1 is positive.

14. Let $\varepsilon > 0$ be given. Is it possible to find an $N > 0$ such that $|a_n - 3| < \varepsilon$ for all $n > N$? Observe that:

$$|a_n - 3| = \left|\frac{3n+2}{n} - 3\right|$$
$$= \left|\frac{3n+2-3n}{n}\right| = \frac{2}{n}.$$

To have $\frac{2}{n} < \varepsilon$ one must have $\frac{2}{\varepsilon} < n$. Therefore, let $N = \frac{2}{\varepsilon}$.

15. The conclusion has two parts:

a. The remainder is a number.

b. The remainder is the number $P(a)$.

Because we want to evaluate the remainder of the division between $P(x)$ and $x - a$, we need to start from the division algorithm. If we are performing long division, we have the following diagram:

$$\begin{array}{r} q(x) \\ x - a \,\overline{)P(x)} \\ \overline{r(x)} \end{array}.$$

The polynomial $q(x)$ represents the quotient, and the polynomial $r(x)$ the remainder of the division. Then we can write $P(x) = (x - a)q(x) + r(x)$. The degree of the remainder must be smaller than the degree of the divisor, $x - a$; otherwise, the division is not complete. Because the degree of $x - a$ is 1, the degree of $r(x)$ must be 0. Thus, $r(x)$ is a number, and we can write $r(x) = r$. Therefore, $P(x) = (x - a)q(x) + r$. This equality is true for all values of the variable x; in particular, we can evaluate it for $x = a$, and we obtain $P(a) = (a - a)q(a) + r = r$. The proof is now complete.

16. There are several ways of proving that these statements are equivalent. We will show that statement 1 implies statement 2, statement 2 implies statement 3, and statement 3 implies statement 1.

Part 1: Statement 1 implies statement 2. Because degree $P(x) \geq$ degree $(x-a)=1$, the polynomial $P(x)$ can be divided by the polynomial $x-a$. Therefore, $P(x)=(x-a)q(x)+r$ (see Exercise 15); however, $P(a)=0$. So $0=P(a)=(a-a)q(a)+r=r$. Thus, the remainder of the division is 0 and $P(x)=(x-a)q(x)$. This means that the polynomial $P(x)$ is divisible by the monomial $x-a$.

Part 2: Statement 2 implies statement 3. By hypothesis, the remainder of the division of $P(x)$ by the polynomial $x-a$ is zero. So, $P(x)=(x-a)q(x)$. By definition, this means that $x-a$ is a factor of $P(x)$.

Part 3: Statement 3 implies statement 1. Because $x-a$ is a factor of $P(x)$, we can write $P(x)=(x-a)q(x)$. Therefore, $P(a)=(a-a)q(a)=0$. This proves that the number a is a root of the polynomial $P(x)$.

17. By hypothesis, the number:

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a) \qquad (*)$$

exists and is finite. We want to show that $\lim_{x\to a}f(x)=f(a)$ or, equivalently, by the properties of limits, that

$$\lim_{x\to a}\Big[f(x)-f(a)\Big]=0$$

To reconstruct the fraction in $(*)$ and thus be able to use the hypothesis, divide and multiply the expression in the brackets by $x-a$. Observe that it is algebraically correct to do so because $x\neq a$; therefore, $x-a\neq 0$. In this way, we obtain:

$$\begin{aligned}\lim_{x\to a}\Big[f(x)-f(a)\Big] &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a).\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to a}\Big[f(x)-f(a)\Big] &= f'(a)\lim_{x\to a}(x-a)\\ &= f'(a)\times 0=0.\end{aligned}$$

Thus

$$\lim_{x \to a} \left[f(x) - f(a) \right] = 0$$

and

$$\lim_{x \to a} f(x) = f(a).$$

18. (This statement can be rewritten as: Let p be a number larger than 2. If p is prime, then p is odd. Because there are infinitely many prime numbers larger than 2, we cannot check directly that they are indeed all odd numbers.) Let p be a number larger than 2. We will assume that p is not odd. Then p must be even. Thus, $p = 2n$ where n is some natural number. Therefore, 2 is a divisor of p, and $2 \neq p$. This contradicts the fact that p is a prime number. (Be careful. Not all odd numbers larger than 2 are prime.)

19. We can show that statement 1 is equivalent to statement 2, and that statement 2 is equivalent to statement 3. Thus, the proof will have four parts:

    Part 1: Statement 1 implies statement 2. Let A^{-1} be the inverse of the matrix A. Let $I_{2\times 2}$ be the 2×2 identity matrix $\left(i.e., I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$. Then $A \times A = I_{2\times 2}$. By the properties of the determinant $\det(A \times A) = \det A \times \det A^{-1} = \det I_{2\times 2} = 1$. Therefore, $\det A \neq 0$.

    Part 2: Statement 2 implies statement 1. We will explicitly find the matrix A^{-1} using the coefficients of the matrix A. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

    We want to construct A^{-1} such that $A \times A = I_{2\times 2}$. From this we obtain a system with four equations in the four unknowns x, y, z, and t:

$$\begin{cases} ax + bz = 1 \\ cx + dz = 0 \\ ay + bt = 0 \\ cy + dt = 1. \end{cases}$$

This system can be separated into two parts:

$$\begin{cases} ax + bz = 1 \\ cx + dz = 0 \end{cases} \quad \text{and} \quad \begin{cases} ay + bt = 0 \\ cy + dt = 1 \end{cases}.$$

The solutions can be found because $\det A = ad - bc \neq 0$. Performing the calculation we obtain:

$$A^{-1} = \begin{pmatrix} \frac{d}{\det A} & \frac{-b}{\det A} \\ \frac{-c}{\det A} & \frac{a}{\det A} \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So A^{-1} is the inverse matrix of A.

Part 3: Statement 2 implies statement 3. The system is formed by the two equations:

$$ax + by = 0 \text{ and } cx + dy = 0.$$

If we solve it, we obtain:

$$(ad - bc)x = 0 \text{ and } (ad - bc)y = 0.$$

Obviously, $x = 0$ and $y = 0$ is a solution. Because $\det A = ad - bc \neq 0$, each one of these two equations has only one solution. Therefore, the system's only solution is $x = 0$ and $y = 0$.

Part 4: Statement 3 implies statement 2. The system is formed by the two equations:

$$ax + by = 0 \text{ and } cx + dy = 0.$$

Because the system has a unique solution, either $a \neq 0$ or $c \neq 0$. Indeed, if $a = 0$ and $c = 0$ the system would have an infinite number of solutions of the form $(x, 0)$, where x is any real number. We assume that $a \neq 0$. Then $x = -by/a$. Substituting this expression into the second equation, we obtain:

$$\frac{(ad - bc)y}{a} = 0.$$

This means that

$$(ad - bc)y = 0.$$

In order for $y = 0$ to be the only solution of this equation, we must have $ad - bc \neq 0$. Therefore, $\det A \neq 0$.

20. Let $\varepsilon > 0$ be given. Is it possible to find a $\delta > 0$ such that $|f(x) - 10| < \varepsilon$ for all x with $|x - 1| < \delta$? Observe that

$$|f(x) - 10| = |3x^2 + 7x - 10| = |x - 1||3x + 10|.$$

How large can the quantity $|3x + 10|$ be, if x is sufficiently close to 1? Start by using values of x closer than 2 units to 1 (this is a completely arbitrary choice); that is, $-1 < x < 3$. Then $-3 < 3x < 9$, and $7 < 3x + 10 < 19$. Therefore, for these values of x we have:

$$|f(x) - 10| = |x - 1||3x + 10| < 19|x - 1|.$$

This quantity is smaller than ε when $|x - 1| < \varepsilon/19$. Thus, choose $\delta = \min\{2, \varepsilon/19\}$. When $|x - 1| < \delta$, it will follow that $|f(x) - 10| < \varepsilon$. Note that different choices of the interval around 1 will produce different choices for δ.

21. This statement will be proved using mathematical induction.

    a. We will show that the formula holds true when $k = 1$, the smallest number we are allowed to use.

$$1^3 = ? = \frac{1^2(1+1)^2}{4}.$$

    Therefore, the equality is true for $k = 1$.

    b. The inductive hypothesis states that the formula holds true for an arbitrary number n; that is,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

    c. We want to prove that:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 = \frac{(n+1)^2[(n+1)+1]^2}{4}.$$

Using the associative property of addition and the inductive hypothesis we obtain:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 &= \left[1^3 + 2^3 + 3^3 + \cdots + n^3\right] + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3. \end{aligned}$$

Performing several simplifications yields:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 &= (n+1)^2\left[\frac{n^2}{4} + (n+1)\right] \\ &= (n+1)^2 \frac{(n^2 + 4n + 4)}{4} \\ &= \frac{(n+1)^2[(n+1)+1]^2}{4}. \end{aligned}$$

Therefore, by the principle of mathematical induction, the given formula holds true for all positive integer numbers.

22. (The two numbers a and b appear in a formula. Therefore, we can try to manipulate the formula to obtain explicit information about them.) Because $ab = (a^2 + 2ab + b^2)/4$, we have $4ab = a^2 + 2ab + b^2$. Thus, $0 = a^2 - 2ab + b^2$. The right-hand side of the equality is equal to $(a-b)^2$. Therefore, we obtain $(a-b)^2 = 0$. This implies that $a - b = 0$. Thus, we can conclude that $a = b$.

23. (We can start working on the given equation, which involves the two numbers a and b, in the hope of obtaining useful clues about them.) From the equality:

$$ab = \frac{(a+b)^2}{2}.$$

we obtain:

$$2ab = a^2 + 2ab + b^2.$$

Therefore,

$$0 = a^2 + b^2.$$

Because a^2 and b^2 are both nonnegative numbers, their sum can be equal to zero if and only if they are both equal to zero (because cancellation is not possible). However, $a^2 = 0$ and $b^2 = 0$ implies $a = b = 0$. Therefore, the statement is true.

24. We are going to prove the given statement using mathematical induction.

a. The smallest number we can use is $k = 2$. We have to add fractions whose denominators are integer numbers between 3 (which corresponds to $k+1$) and 4 (which corresponds to $2k$). Therefore, the left-hand side of the equation becomes

$$\frac{1}{3}+\frac{1}{4}=\frac{7}{12}.$$

Because $7/12 > 1/2$, the statement is true in this case.

b. We assume that the inequality is true for an arbitrary number n. Thus,

$$\frac{1}{n+1}+\frac{1}{(n+1)+1}+\cdots+\frac{1}{2n}>\frac{1}{2}.$$

c. We need to prove that the inequality holds true for $n+1$. We will add fractions with denominators between $(n+1)+1$ and $2(n+1)$. So, we want to prove that

$$\frac{1}{(n+1)+1}+\frac{1}{(n+1)+2}+\cdots+\frac{1}{2n+1}+\frac{1}{2(n+1)}>\frac{1}{2}.$$

One thing to notice is that the largest denominator of the fractions in the inductive hypothesis is $n+1$, while the largest denominator in this step is $n+2$. Thus, to make the inequality in the inductive hypothesis and the left-hand side of the inequality to be proved start with fractions having the same denominator, we could rewrite the inductive hypothesis as:

$$\frac{1}{n+2}+\frac{1}{(n+2)+1}+\cdots+\frac{1}{2n}>\frac{1}{2}-\frac{1}{n+1}.$$

Using the associative property of addition and the rewritten inductive hypothesis, we obtain

$$\begin{aligned}
&\frac{1}{(n+1)+1}+\frac{1}{(n+1)+2}+\cdots+\frac{1}{2n+1}+\frac{1}{2(n+1)}\\
&\quad=\left[\frac{1}{(n+1)+1}+\frac{1}{(n+1)+2}+\cdots+\frac{1}{2n}\right]+\frac{1}{2n+1}+\frac{1}{2(n+1)}\\
&\quad>\left(\frac{1}{2}-\frac{1}{n+1}\right)+\frac{1}{2n+1}+\frac{1}{2(n+1)}\\
&\quad=\frac{1}{2}+\frac{1}{(2n+1)(2n+2)}>\frac{1}{2}.
\end{aligned}$$

The statement is now proved. Therefore, by the principle of mathematical induction, the inequality holds true for all integers $k \geq 2$.

25. Because a is a multiple of b, we can write $a = bn$ for some integer n. Because b is a multiple of c, we can write $b = cm$ for some integer m. We will now combine this information to find a direct relation between a and c. Thus, we obtain $a = bn = (cm)n = c(mn)$. Since the number mn is an integer (as it is the product of two integers), we can conclude that a is a multiple of c.

26. The proof of this statement has two components—namely, the proofs of the following statements:

    a. If p is a nonzero rational number, then its reciprocal is a rational number,

    b. If the reciprocal of a nonzero number p is a rational number, then the number itself is a rational number.
    The reciprocal of a nonzero number p is the number q such that $pq = 1$.

    Proof of part a. Because the number p is rational and nonzero, we can write $p = a/b$, where a and b are relatively prime numbers, both not equal to zero. Therefore, $(a/b)q = 1$. If we multiply both sides of the equation by b and divide them by a, we obtain $q = b/a$. This means that q is a rational number.

    Proof of part b. Because the inverse of the nonzero number p, which we will indicate with p^{-1}, is a rational number, we can write it as $p^{-1} = c/d$, where c and d are relatively prime numbers, both not equal to zero. Thus,

$$pp^{-1} = p\frac{c}{d} = 1.$$

    If we multiply the equality by d and divide it by c, we obtain $p = d/c$, where c and d are relatively prime numbers, both not equal to zero. Thus, p is a rational number. (*Note:* In this proof we assume that both p and q are in reduced form, by stating that a and b are relatively prime, and c and d are relatively prime. While it is correct to make these assumptions, in this case there is no need for them. The proof is still correct if the "relatively prime" requirement is removed.)

27. Let $\varepsilon > 0$ be given. Is there an $N > 0$ such that $|a_n - 0| < \varepsilon$ for all $n > N$? Observe that:

$$|a_n - 0| = \left|\left(-\frac{1}{2}\right)^n - 0\right| = \left|\left(-\frac{1}{2}\right)^n\right| = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}.$$

In order for $1/2^n$ to be smaller than ε, we must have $1/\varepsilon < 2^n$ or $n > (\ln 1/\varepsilon)/\ln 2$. To be sure that $N > 0$, choose $N = \max\{1, (\ln 1/\varepsilon)/\ln 2\}$.

28. Because a, b, and c are three consecutive integers, without loss of generality we can assume that a is the smallest of them and write $b = a + 1$ and $c = a + 2$. Then $a + b + c = a + (a + 1) + (a + 2) = 3a + 3 = 3(a + 1) = 3b$. Because b is an integer number, the equality proves that $a + b + c$ is divisible by 3. *Note:* We cannot use proof by induction for this statement because the three numbers could be negative. Therefore, there is no smallest number for which to check that the statement is true. The statement "Let a, b, and c be three consecutive positive integer numbers; then 3 divides the sum $a + b + c$" could be proved by induction. Try this method, and see what happens. (Does this result relate in any way to finding the average of three consecutive integer numbers?)

29. The proof is constructed by induction.

    a. We need to check whether the statement is true for $k = 0$. Because $k^3 - k = 0 - 0 = 0$, and 0 is divisible by 3, the statement is indeed true.

    b. Let us assume that the statement is true for a generic number $n \geq 1$; that is, $n^3 - n = 3p$ for some integer number p.

    c. We now need to prove that $(n + 1)^3 - (n + 1) = 3t$ for some integer number t. Performing some algebraic steps we obtain:

$$\begin{aligned}(n + 1)^3 - (n + 1) &= n^3 + 3n^2 + 3n - n \\ &= (n^3 - n) + 3(n^2 + n).\end{aligned}$$

The number $n^2 + n$ is an integer. Call it q. Then, using the inductive hypothesis yields:

$$(n + 1)^3 - (n + 1) = 3p + 3q = 3(p + q).$$

Because the number $p + q$ is an integer, we have proved that the statement is true. Therefore, by the principle of mathematical induction the statement is true for all whole numbers. (*Note:* There is another way of proving this statement without using mathematical induction. Indeed, $n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1)$. The three numbers n, $n + 1$, and $n - 1$ are consecutive. So, to complete the proof, we could prove that one of them is divisible by 3. In Exercise 28 we proved that the *sum* of three consecutive integers is

divisible by 3. Part of this exercise is to prove that the *product* of three consecutive numbers is also divisible by 3.)

30. We will prove this statement using its contrapositive. We will assume that there exists at least one sequence $\{c_n\}_{n=1}^{\infty}$ such that $c_n > 0$ for all n and $L < 0$. By definition of limit of a sequence, for every $\varepsilon > 0$ there exists an N such that $|c_n - L| < \varepsilon$ for all $n > N$. This is equivalent to stating $L - \varepsilon < c_n < L + \varepsilon$ for all $n > N$. Consider the positive number $\varepsilon_0 = -L/2$. Because L is the limit of the sequence, there exists a corresponding M_{ε_0} such that $L - (L/2) < c_n < L + (-L/2)$, for all $n > M_{\varepsilon_0}$. Therefore,

$$\frac{3L}{2} < c_n < \frac{L}{2}, \quad \text{for all} \quad n > M_{\varepsilon_0}.$$

The numbers $3L/2$ and $L/2$ are negative. So we reached the conclusion that $c_n < 0$ for all $n > M_{\varepsilon_0}$. This contradicts the statement that $c_n > 0$ for all n. Therefore, the limit of a sequence of positive numbers cannot be negative. We cannot say that the limit of a sequence of positive numbers has to be positive, because it could be 0. Indeed, consider the sequence $\{1/n\}_{n=1}^{\infty}$. Every term of the sequence is positive, but the limit of this sequence is 0.

31. To prove that $\lim_{x \to 3} f(x) = \sqrt{3}$, one needs to prove that for every given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - \sqrt{3}| < \varepsilon$ for all x with $|x - 3| < \delta$. Observe that

$$\left|f(x) - \sqrt{3}\right| = \left|\sqrt{x} - \sqrt{3}\right| = \left|\frac{(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})}{(\sqrt{x} + \sqrt{3})}\right|.$$

Therefore, $|f(x) - \sqrt{3}| = |x - 3| \frac{1}{\sqrt{x} + \sqrt{3}}$. How large is the factor $\frac{1}{\sqrt{x} + \sqrt{3}}$ for x relatively close to 3? If $|x - 3| < 1$ (this is an arbitrary choice), then $2 < x < 4$ and $\sqrt{2} < \sqrt{x} < \sqrt{4} = 2$. Thus, $\sqrt{2} + \sqrt{3} < \sqrt{x} + \sqrt{3} < 2 + \sqrt{3}$, or $2\sqrt{2} < \sqrt{x} + \sqrt{3} < 4$. Therefore, $(1/(2\sqrt{2})) > (1/(\sqrt{x} + \sqrt{3})) > (1/4)$, and $|f(x) - \sqrt{3}| = |x - 3| \frac{1}{\sqrt{x} + \sqrt{3}} < \frac{1}{2\sqrt{2}} |x - 3|$. The value of this expression will be smaller than the given $\varepsilon > 0$ if $\frac{1}{2\sqrt{2}} |x - 3| < \varepsilon$; that is, if $|x - 3| < 2\sqrt{2}\varepsilon$. Thus, choose $\delta = \min\{1, 2\sqrt{2}\varepsilon\}$.

32. *Existence*: Because $ad - bc \neq 0$, at least two of these four numbers are not equal to 0. Without loss of generality, we will assume $a \neq 0$. From the first equation we obtain $x = (e - by)/a$. We can substitute this formula for x into the second equation to obtain

$y = (af - ce)/(ad - bc)$. If we substitute this representation of y into the formula for x, we find that $x = (de - bf)/(ad - bc)$. The two fractions are well defined because $ad - bc \neq 0$. Therefore, we have found a solution of the given system. (Check what would happen for other combinations of nonzero coefficients, in addition to $ad - bc \neq 0$.)

Uniqueness: The solution just found is unique because the values of x and y are uniquely determined by the two equations found above.

Note: The uniqueness of the solution can be established in another way, which is considerably longer. Assume that (x_1, y_1) and (x_2, y_2) are two solutions. Therefore,

$$\begin{cases} ax_1 + by_1 = e \\ cx_1 + dy_1 = f \end{cases} \quad \text{and} \quad \begin{cases} ax_2 + by_2 = e \\ cx_2 + dy_2 = f \end{cases}.$$

Then

$$\begin{aligned} ax_1 + by_1 &= ax_2 + by_2 \\ cx_1 + dy_1 &= cx_2 + dy_2 \end{aligned}.$$

This is equivalent to:

$$\begin{aligned} a(x_1 - x_2) &= b(y_1 - y_2) \\ c(x_1 - x_2) &= d(y_1 - y_2) \end{aligned}.$$

Therefore, if $a \neq 0$, $x_1 - x_2 = b(y_1 - y_2)/a$. Substituting this expression into the second equation yields $(y_1 - y_2)(ad - bc) = 0$. Because $ad - bc \neq 0$, this equality implies $y_1 - y_2 = 0$; that is, $y_1 = y_2$. From this conclusion, we obtain that $x_1 = x_2$, as well. Thus, the two solutions coincide.

33. This statement will be proved by induction.

    a. The smallest number we can use is $k = 6$. In this case, we obtain $2^6 = 64 > (6 + 1)^2 = 49$. Thus, the inequality is true in this case.

    b We will now assume that $2^n > (n + 1)^2$.

    c. We need to prove that $2^{n+1} > [(n + 1) + 1]^2$. Using algebra rules and the inductive hypothesis, we obtain $2^{n+1} = 2 \times 2^n > 2(n + 1)^2 = 2n^2 + 4n + 2$. We need to see how the expression we just obtained compares with the right-hand side of the inequality we want to

obtain, namely $[(n+1)+1]^2$. If we evaluate $[(n+1)+1]^2$ we obtain $[(n+1)+1]^2 = n^2+4n+4$. Therefore, we have to compare the expressions $2n^2+4n+2$ and n^2+4n+4. We will do so by calculating their difference: $(2n^2+4n+2)-(n^2+4n+4) = n^2-2$. Because $n^2 \geq 36$, $n^2-2>0$.
This proves that $(2n^2+4n+2)>(n^2+4n+4)$. If we now list all the steps just performed altogether, we have:

$$2^{n+1} = 2 \times 2^n > 2(n+1)^2 = 2n^2+4n+2$$
$$> n^2+4n+4 = [(n+1)+1]^2.$$

The inequality is therefore true. So, by the principle of mathematical induction, the inequality will hold true for all integers $k \geq 6$.

34. Because this is an existence statement, it is enough to find one number (not necessarily an integer) such that $2^k > (k+1)^2$. Consider $k=6$.

35. The statement does not seem to be true. We can try to prove that it is false by providing a counterexample. Consider $t=1$ and $q=1/2$. Then $t+q=3/2$, and this is not an irrational number. (We can indeed prove that the statement is false for every two rational numbers. We can prove that the sum of two rational numbers is always a rational number. Indeed, if t and q are two rational numbers, we can write $t=a/b$, where a and b are relatively prime integers, and $b \neq 0$; and $q=c/d$, where c and d str relatively prime integers, and $d \neq 0$. Then $bd \neq 0$, and $t+q=(ad+bc)/bd$. Therefore, $t+q$ is a well-defined rational number because $ad+bc$ and bd are both integers and $bd \neq 0$.)

36. This is an existence statement. To prove that it is true we only need to exhibit three consecutive integer numbers whose sum is a multiple of 3. Consider 3, 4, and 5. Then $3+4+5=12$, which is a multiple of 3. The fact that this statement is true for any three consecutive integer numbers (see Exercise 28) is irrelevant. It just makes it very easy to find an example.

37. The statement seems false; therefore, we will search for a counter-example. Consider $n=5$. Then 5 is a multiple of itself, but $5^2 = 25$, which is not a multiple of 125.

38. (We cannot use proof by induction because we do not know what the smallest number is that can be used to establish the base case.) We are going to prove that $n^2+n = 2t$ for some integer number t. Using factorization we can write $n^2+n = n(n+1)$. If n is an even number, then $n=2q$ for some integer number q. Thus, $n^2+n = n(n+1) = 2[q(n+1)]$. Because the number $q(n+1)$ is an integer, this proves that

$n^2 + n$ is an even number. If n is an odd number, then $n = 2k + 1$ for some integer number k. Thus,

$$\begin{aligned} n^2 + n &= (2k + 1)[(2k + 1) + 1] \\ &= (2k + 1)(2k + 2) = 2[(2k + 1)(k + 1)]. \end{aligned}$$

Because the number $(2k + 1)(k + 1)$ is an integer, this proves that $n^2 + n$ is an even number. Therefore, the statement is true.

39. We will prove this statement by induction.

    a. Let us check if the inequality holds for $k = 6$. In this case, we obtain $6! = 720 > 216 = 6^3$.

    b. Assume that $n! > n^3$.

    c. We have to prove that $(n + 1)! > (n + 1)^3$. By the properties of factorials, $(n + 1)! = (n + 1)n!$. If we use this fact and the inductive hypothesis, we obtain $(n + 1)! = (n + 1)n! > (n + 1)n^3$. We will now use this inequality and other algebraic properties of inequalities to obtain the expression $n^3 + 3n^2 + 3n + 1 = (n + 1)^3$:

$$\begin{aligned} (n + 1)! &= (n + 1)n! > (n + 1)n^3 \\ &\geq (6 + 1)n^3 = n^3 + 6n^3 \\ &> n^3 + 6n^2 = n^3 + 3n^2 + 3n^2 \\ &= n^3 + 3n^2 + 3n \times n > n^3 + 3n^2 + 3 \times 6 \times n \\ &> n^3 + 3n^2 + 3n + n > n^3 + 3n^2 + 3n + 1 \\ &= (n + 1)^3. \end{aligned}$$

    Therefore, by the principle of mathematical induction, the original statement is true for all integers $k \geq 6$.

40. Assume that $\lim_{n\to\infty}\{(-1)^n \frac{1}{5}\} = L$, where L is a real number. Then, for every $\varepsilon > 0$, there exists an $N > 0$ such that $|(-1)^n \frac{1}{5} - L| < \varepsilon$ for all $n > N$. This implies that $-\varepsilon < L - (-1)^n(1/5) < \varepsilon$; that is, $-\varepsilon + (-1)^n(1/5) < L < \varepsilon + (-1)^n \frac{1}{5}$. Because these inequalities will hold true for all $n > N$, they will hold true for odd and even values of n. Thus, one has to consider the following two sets of inequalities:

$$-\varepsilon + \frac{1}{5} < L < \varepsilon + \frac{1}{5}$$

and

$$-\varepsilon - \frac{1}{5} < L < \varepsilon - \frac{1}{5}.$$

Because ε can be any positive number, they will have to hold true even when $\varepsilon = 1/5$. In this case, the first set of inequalities will yield $0 < L$ and the second $L < 0$. Clearly this is impossible. Therefore, the limit of the sequence does not exist.

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Other Books on the Subject of Proofs and Mathematical Writing

The majority of the following books have parts that concentrate on logic and the construction of proofs, as well as parts that introduce new mathematical concepts. Some of them are considerate to be classics, while others are quite more recent.

Barnier, W. and Feldman, N., *Introduction to Advanced Mathematics*, Prentice Hall, New York, 2000. (Read Chapters 1 and 2.)

Bloch, E. D., *Proofs and Fundamentals: A First Course in Abstract Mathematics*, Birkhäuser, Basel, 2000. (Read the first three chapters.)

Chartrand, G., Polimeni A. D., and Zhang, P., *Mathematical Proofs: A Transition to Advanced Mathematics*, Addison-Wesley, Reading, MA, 2002.

Copi, I. M., *Introduction to Logic*, Prentice Hall, New York, 1998.

D'Angelo, J. and West D., *Mathematical Thinking: Problem-Solving and Proofs*, 2nd ed., Prentice Hall, New York, 2000. (Read Part 1: Elementary Concepts.)

Eccles, P. J., *An Introduction to Mathematical Reasoning: Numbers, Sets and Functions*, Cambridge University Press, Cambridge, U.K., 1998.

Enderton, H., *A Mathematical Introduction to Logic*, 2nd ed., Academic Press, San Diego, CA, 2000.

Exner, G. R., *An Accompaniment to Higher Mathematics*, Undergraduate Texts in Mathematics, Springer-Verlag, Berlin/New York, 1996.

Franklin, J. and Dauod, A., *Proof in Mathematics: An Introduction.* Quakers Hill Press, Sydney, 2001.

Garnier, R. and Taylor, J., *100% Mathematical Proof*, John Wiley & Sons, New York, 1996.

Hodel, R. E., *An Introduction to Mathematical Logic*, PWS-Kent, Boston, MA, 1995.

Hermes, H., *Introduction to Mathematical Logic*, Springer-Verlag, Berlin/New York, 1973. (Read Part I: Introduction.)

Lucas, J. F., *Introduction to Abstract Mathematics*, Ardsley House Publications, 1990. (Read Chapters 1 and 2.)

Morash, R. P., *Bridge to Abstract Mathematics*, McGraw Hill, New York, 1991. (Read Chapters 1 to 5.)

Penner, R. C., *Discrete Mathematics: Proof Techniques and Mathematical Structures*, World Scientific, Singapore, 1999.

Polya, G., *How To Solve It*, Princeton University Press, Princeton, NJ, 1988.

Polya, G., *Mathematical Discovery*, John Wiley & Sons, New York, 1962.

Rodgers, N., *Learning To Reason: An Introduction to Logic, Sets and Relations*, John Wiley & Sons, New York, 2000. (Read the section "Writing Our Reasoning.")

Rotman, J. J., *A Journey into Mathematics: An Introduction to Proofs*, Prentice Hall, New York, 1997.

Schumacher, C., *Chapter Zero. Fundamental Notions of Abstract Mathematics*, Addison-Wesley, Reading, MA, 1996. (Read Chapters 1 and 2.)

Schwartz, D., *Conjecture and Proof: An Introduction to Mathematical Thinking*, Saunders College Publishing, Philadelphia, PA, 1997. (Read Chapters 2 and 3.)

Solow, D., *How To Read and Do Proofs*, John Wiley & Sons, New York, 1990.

Solow, D., *Reading, Writing and Doing Mathematical Proofs: Proof Techniques for Advanced Mathematics*, Dale Seymour Publications, 1984.

Stolyar, A. A., *Introduction to Mathematical Logic*, Dover, New York, 1984. (Read the first two sections.)

Suppes, P., *Introduction to Logic.* Dover, New York, 1999. (Chapter 7 is quite interesting.)

Velleman, D. J., *How To Prove It: A Structured Approach*, Cambridge University Press, Cambridge, U.K., 1994.

Whitehead, A. N., *Introduction to Mathematics*, Oxford University Press, London, 1948.

Wickelgren, W. A., *How To Solve Problems*, Dover, New York, 1995.

Wolfe, R. S., *Proof, Logic, Conjecture: The Mathematical Toolbox*, W. H. Freeman, New York, 1998. (Read Chapters 1 to 4.)

An interesting and useful book for a quick review of mathematical terms is the well-known *Mathematics Dictionary* by Glenn James and Robert C. James (Chapman & Hall, 1992).
A book that goes to the roots of mathematical words is *The Words of Mathematics: An Etymological Dictionary of Mathematical Terms Used in English*, by Steven Schwartzman (The Mathematical Association of America, 1994).

The following books offer suggestions on writing mathematical material:

Alley, M., *The Craft of Scientific Writing*, Springer-Verlag, Berlin/New York, 1996.
Countryman, J., *Writing To Learn Mathematics*, Heinemann, Portsmouth, NH, 1992.
Gerver, R., *Writing Research Papers: Enrichment for Math Enthusiasts*, Key Curriculum Press, 1997.
Gillman, L., *Writing Mathematics Well*, Mathematical Association of America, Washington, D.C., 1987.
Higham, N. J., *Handbook of Writing for the Mathematical Sciences*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1993.
Knuth, D. E., Larrabee, T., and Roberts, P. M., *Mathematical Writing*, Mathematical Association of America, Washington, D.C., 1989.
Levine, A., *Discovering Higher Mathematics: Four Habits of Highly Effective Mathematicians*, Academic Press, San Diego, CA, 2000. (This is a book that focuses on the processes of experimentation, conjecture, proof, and generalization, by using material form number theory, discrete mathematics, and combinatorics. The author also addresses some issues in writing mathematics.)

A multitude of books are available on how to use the tools of logic to solve problems. Here is a brief (and thus incomplete) list of them:

Daepp, U. and Garkin, P., *Reading, Writing and Proving: A Closer Look at Mathematics*, Springer-Verlag, Berlin/New York, 2003.
Engel, A., *Problem Solving Strategies*, Springer-Verlag, Berlin/New York, 1998.
Larson, L.C., *Problem Solving Through Problems*, Springer-Verlag, Berlin/New York, 1983.
Lozansky, E. and Rousseau, C., *Winning Solutions*, Problem Books in Mathematics, Springer-Verlag, Berlin/New York, 1996.
Vakil, R., *A Mathematical Mosaic: Patterns and Problem Solving*, Brendan Kelly Publishing, 1997.

Williams, K. S. and Hardy, K., *The Red Book of Mathematical Problems*, Dover, New York, 1996.
Williams, K. S., Hardy, K. *The Green Book of Mathematical Problems*, Dover, New York, 1997.
Zeitz, P., *The Art and Craft of Problem Solving*, John Wiley & Sons, New York, 1999.

A book to enjoy after mastering the basics of mathematical proofs is *Proofs from the Book*, 3rd ed., by M. Aigner and G. Ziegler (Springer-Verlag, 2003).

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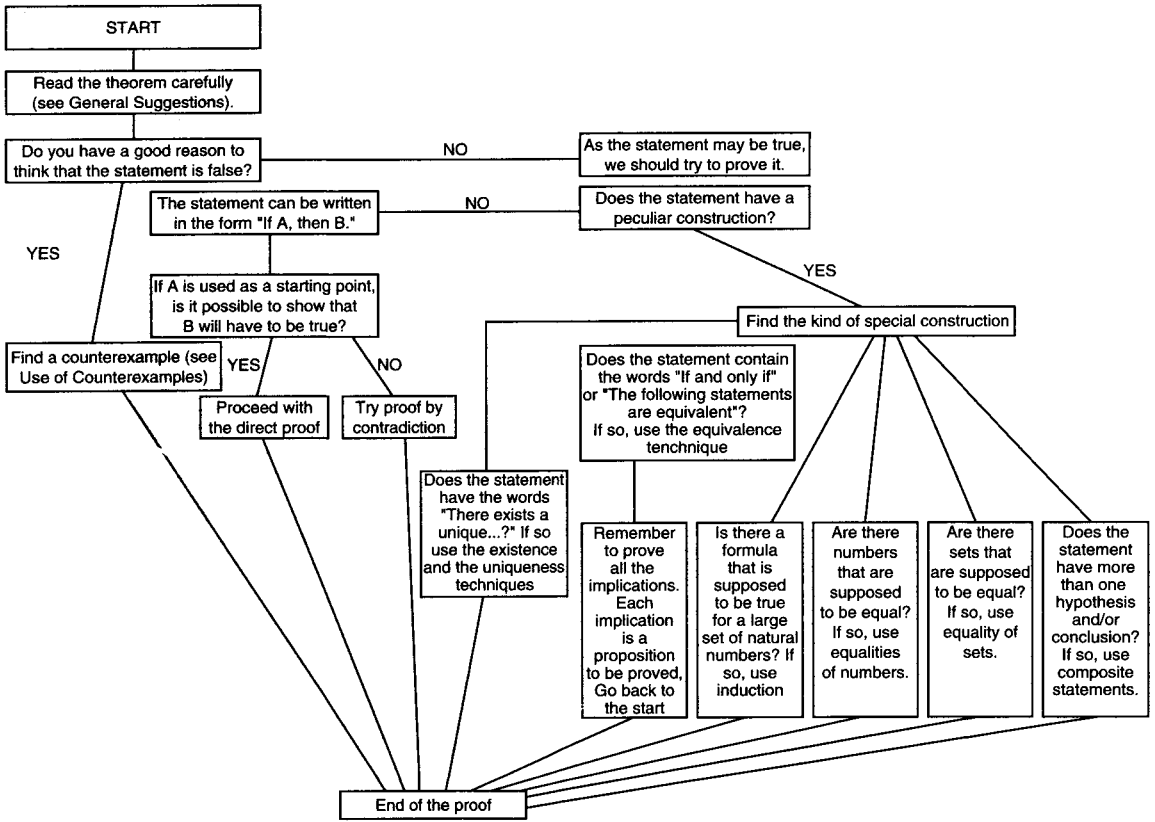
A Guide to Selecting a Method of Proof


START
Read the theorem carefully (see General Suggestions).
Do you have a good reason to think that the statement is false?
NO
As the statement may be true, we should try to prove it.
YES
The statement can be written in the form "If A, then B."
NO
Does the statement have a peculiar construction?
YES
If A is used as a starting point, is it possible to show that B will have to be true?
Find the kind of special construction
Find a counterexample (see Use of Counterexamples)
YES
NO
Proceed with the direct proof
Try proof by contradiction
Does the statement contain the words "If and only if" or "The following statements are equivalent"? If so, use the equivalence tenchnique
Does the statement have the words "There exists a unique...?" If so use the existence and the uniqueness techniques
Remember to prove all the implications. Each implication is a proposition to be proved, Go back to the start
Is there a formula that is supposed to be true for a large set of natural numbers? If so, use induction
Are there numbers that are supposed to be equal? If so, use equalities of numbers.
Are there sets that are supposed to be equal? If so, use equality of sets.
Does the statement have more than one hypothesis and/or conclusion? If so, use composite statements.
End of the proof